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HOMOLOGY AND COHOMOLOGY FOR CLOSURE SPACES

defendida por

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A mi papá, mi mamá y mi hermano.
Sin el apoyo, confianza y seguridad constante de ellos no sería la persona que soy hoy en día.

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Introduction

Topological Data Analysis (TDA) is a recent development in data analysis where the focus is on the geometry of a data sample, and whose primary tools come from algebraic topology. In TDA, the main goal is to infer geometric and topological information of a topological space using data sampled from the space. Nevertheless, there is an obstacle when dealing with finite points embedded in a metric space. The natural way to endow a topology on a finite set is to give it the subspace topology, which, in the case of metric spaces, coincides with the discrete topology. A common approach in TDA avoids this problem by ”approximating” the space with balls centered at each of the sampled points. Then, using a range of different radii in order to induce a filtration, one computes the so called ”persistent homology” [7].

While this method is adequate when considering one space at the time, unfortunately the process is not functorial. That is, given a map between two different sets of finite points, there is no canonical map between the corresponding unions of balls, nor between the corresponding persistent homology groups. Without this functorial property, many tools such as the Mayer-Vietoris sequence and homotopy invariance are lost in this setting.

There has also been increasing interest in computing homology at a fixed scale. Several computations are accomplished in [11], [1], and [3]. Since neither the appropriate exact nor spectral sequences have been developed in this setting, the techniques in these papers are built directly on the definition, which makes these homologies hard to compute. The papers implicitly encode the scale into the homology, which also fails to preserve the functoriality (from **Top** to **Ab**).

In this thesis, we will develop Čech homology and cohomology theories for closure spaces, also known as Čech spaces. Closure spaces, as with topological spaces, are uniquely determined by a neighborhood system at each point, but with closure spaces we can arrange for every neighborhood to contain a ball of non-zero radius. This gives us a way to encode a scale into the space itself. We will see that encoding the scale to the space instead of the homology will be the key to achieving functoriality. We will go in more detail on closure spaces in general in the first chapter using the book [2] as a guide. We then construct Čech homology and cohomology. Our primary goal for these theories is for them to have functoriality, excision and homotopy invariance properties, but we check all the Eilenberg-Stenrod axioms for (co)homology. Finally, once the Eilenberg-Steenrod axioms are established, we derive a Mayer-Vietoris theorem for Čech cohomology in the context of closure spaces in a similar way as described in [5]. A systematic approach to the algebraic topology of Čech spaces was started in [9], although the possibility is mentioned sporadically in the literature [8].

Chapter 1

Closure Spaces

In this chapter we will introduce the basic definitions and properties of closure spaces, which will be used in the following chapters. These can be found in the chapter III of [2].

Definition 1.1. *A closure space* (X, c) is a set X along with an map $c : \mathscr{P}(X) \to \mathscr{P}(X)$ that satisfies:

C1) $c(\emptyset) = \emptyset$

C2) For all $U \subset X$, $U \subset c(U)$

C3) For all $U_1, U_2 \subset X$, $c(U_1 \cup U_2) = c(U_1) \cup c(U_2)$

We say that c is *the closure operator of* X. If there is no ambiguity, we will refer to c as the *closure of* X.

Definition 1.2. *The interior operator of* X is an map $\iota : \mathscr{P}(X) \to \mathscr{P}(X)$ defined by

$$i(U) := X \setminus c(X \setminus U)$$

for all $U \subset X$.

Observation 1. The interior operator i satisfies the following properties, derived from the closure axioms (C1), (C2) and (C3):

I1) $i(X) = X$
By definition and (C1)

$$i(X) = X \setminus c(X \setminus X) = X \setminus c(\emptyset) = X \setminus \emptyset = X$$

I2) For all $U \subset X$, then $i(U) \subset U$
Let $U \subset X$. Using (C2) we have that $X \setminus U \subset c(X \setminus U)$ and so

$$i(U) = X \setminus c(X \setminus U) \subset X \setminus (X \setminus U) = U$$

I3) For all $U_1, U_2 \subset X$, we have $i(U_1 \cap U_2) = i(U_1) \cap i(U_2)$
Let $U_1, U_2 \subset X$. Using De Morgan's laws we have that $X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2)$. It follows, using (C3), that

$$\begin{aligned} i(U_1 \cap U_2) &= X \setminus c(X \setminus (U_1 \cap U_2)) \\ &= X \setminus c((X \setminus U_1) \cup (X \setminus U_2)) \\ &= X \setminus [c((X \setminus U_1)) \cup c((X \setminus U_2))] \\ &= [X \setminus c((X \setminus U_1))] \cap [X \setminus c((X \setminus U_2))] \\ &= i(U_1) \cap i(U_2) \end{aligned}$$

Lemma 1.1. *Given a set X and an operator $i : \mathscr{P}(X) \to \mathscr{P}(X)$ that satisfies (I1), (I2), (I3), then there is a unique closure operator c : $\mathscr{P}(X) \to \mathscr{P}(X)$ such that i is the interior operator in the closure space (X, c).*

Proof. If c_1 and c_2 are closure operators such that i is the interior operator of both c_1 and c_2, then for any $U \subset X$ we have that

$$i(X \setminus U) = X \setminus c_1(U) = X \setminus c_2(U)$$

Thus, we have that $c_1(U) = c_2(U)$, which proves that uniqueness of the closure operator.

Now, for any $U \subset X$, define

$$c(U) = X \setminus i(X \setminus U)$$

To prove that c is a closure operator, we have to show that it satisfies (C1), (C2), and (C3).

- Proof of (C1)
  Note that

  $$c(\emptyset) = X \setminus i(X \setminus \emptyset) = X \setminus i(X) = X \setminus X = \emptyset$$

  Therefore, c satisfies (C1).

- Proof of (C2)
  For any $A \subset X$, using (I2), we have that $i(X \setminus A) \subset X \setminus A$; therefore,

  $$A = X \setminus (X \setminus A) \subset X \setminus i(X \setminus A) = c(A)$$

- Proof of (C3)

Using (I3) we have that

$$\begin{aligned}
c\left(U_1 \cup U_2\right) &= X \setminus i\left(X \setminus \left(U_1 \cup U_2\right)\right) \\
&= X \setminus i\left(\left(X \setminus U_1\right) \cap \left(X \setminus U_2\right)\right) \\
&= X \setminus \left(i\left(X \setminus U_1\right) \cap i\left(X \setminus U_2\right)\right) \\
&= \left(X \setminus i\left(X \setminus U_1\right)\right) \cup \left(X \setminus i\left(X \setminus U_2\right)\right) \\
&= c\left(U_1\right) \cup c\left(U_2\right)
\end{aligned}$$

We conclude that indeed c is indeed a closure operator. □

Definition 1.3. A function $f : (X, \mathrm{c}_X) \to (Y, \mathrm{c}_Y)$ between closure spaces is said to be *continuous* if

$$f(c_X\left(A\right)) \subset c_Y\left(f(A)\right)$$

for all $A \subset X$.

Proposition 1.2. *Given a function $f : (X, c_X) \to (Y, c_Y)$ between closure spaces the following are equivalent:*

1) f is continuous.

2) For all $B \subset Y$, $c_X\left(f^{-1}\left(B\right)\right) \subset f^{-1}\left(c_Y\left(B\right)\right)$.

3) For all $B \subset Y$, $f^{-1}\left(i_Y\left(B\right)\right) \subset i_X\left(f^{-1}\left(B\right)\right)$, with i_X and i_Y are the interior operators for X and Y, respectively.

Proof.

$1) \Rightarrow 2)$

Suppose f is continuous. Given $B \subset Y$ define $A := f^{-1}\left(B\right)$. Remember that $f(A) = f(f^{-1}\left(B\right)) \subset B$. Using the continuity of f we conclude that

$$f(c_X\left(A\right)) \subset c_Y\left(f(A)\right) \subset c_Y\left(B\right).$$

Therefore,

$$\begin{aligned}
c_X\left(f^{-1}\left(B\right)\right) &= c_X\left(A\right) \\
&\subset f^{-1}\left(f\left(c_X\left(A\right)\right)\right) \\
&\subset f^{-1}\left(c_Y\left(B\right)\right)
\end{aligned}$$

2) $\Rightarrow$ 3)

Given $B \subset Y$, by defintion

$$\begin{aligned}
f^{-1}\left(i_Y\left(B\right)\right) &= f^{-1}\left(Y \setminus c_Y\left(Y \setminus B\right)\right) \\
&= f^{-1}\left(Y\right) \setminus f^{-1}\left(c_Y\left(Y \setminus B\right)\right) \\
&= X \setminus f^{-1}\left(c_Y\left(Y \setminus B\right)\right) \\
&\subset X \setminus c_X\left(f^{-1}\left(Y \setminus B\right)\right) \\
&= X \setminus c_X\left(f^{-1}\left(Y\right) \setminus f^{-1}\left(B\right)\right) \\
&= X \setminus c_X\left(X \setminus f^{-1}\left(B\right)\right) \\
&= i_X\left(f^{-1}\left(B\right)\right)
\end{aligned}$$

3) $\Rightarrow$ 2)

Given $B \subset Y$,

$$\begin{aligned}
c_X\left(f^{-1}\left(B\right)\right) &= X \setminus i_X\left(X \setminus f^{-1}\left(B\right)\right) \\
&= X \setminus i_X\left(f^{-1}\left(Y\right) \setminus f^{-1}\left(B\right)\right) \\
&= X \setminus i_X\left(f^{-1}\left(Y \setminus B\right)\right) \\
&\subset X \setminus f^{-1}\left(i_Y\left(Y \setminus B\right)\right) \\
&= f^{-1}\left(Y\right) \setminus f^{-1}\left(i_Y\left(Y \setminus B\right)\right) \\
&= f^{-1}\left(Y \setminus i_Y\left(Y \setminus B\right)\right) \\
&= f^{-1}\left(c_Y\left(B\right)\right)
\end{aligned}$$

2) $\Rightarrow$ 1)

Let $A \subset X$, we have that

$$c_X\left(A\right) \subset c_X\left(f^{-1}\left(f\left(A\right)\right)\right) \subset f^{-1}\left(c_Y\left(f\left(A\right)\right)\right).$$

Therefore

$$f\left(c_X\left(A\right)\right) \subset f\left(f^{-1}\left(c_Y\left(f\left(A\right)\right)\right)\right) \subset c_Y\left(f\left(A\right)\right)$$

$\square$

Definition 1.4. Given a closure space (X, c), and a subset $A \subset X$. A subset $U \subset X$ is called a *neighborhood of A* if

$$A \subset i\left(U\right)$$

The collection of all neighborhoods of A is called *the neighborhood system of A in X*, and we denote it by $\mathcal{N}(A)$. If $A = \{x\}$, i.e., it's just the set of a single point $x \in X$, then its neighborhood system will be denoted by $\mathcal{N}_x$.

Definition 1.5. A *filter* (definition 12.B2, [2]) on a set X is a non empty collection $\mathscr{F}$ of subsets of X such that

- If $U \in \mathscr{F}$ and $V \subset X$ such that $U \subset V$, then $V \in \mathscr{F}$.
- If $U, V \in \mathscr{F}$, then $U \cap V \in \mathscr{F}$.

If $\emptyset \notin \mathscr{F}$, we say that $\mathscr{F}$ is a *proper filter*.

We say that a subset $\mathscr{B} \subset \mathscr{F}$ is a *base of the filter* $\mathscr{F}$ if for all $U \in \mathscr{F}$ there is $V \in \mathscr{B}$ such that $V \subset U$.

If $\gamma \subset \mathscr{P}(X)$ is a non empty collection of subsets of X such that all finite intersections form a base of the filter $\mathscr{F}$, we say that γ is a *subbase of* $\mathscr{F}$.

Proposition 1.3. *A non empty collection of subsets $\mathcal{B} \subset \mathscr{P}(X)$ is a base for a filter on X if and only if for all $U, V \in \mathcal{B}$ there is $W \in \mathcal{B}$ such that $W \subset U \cap V$.*

Proof. Suppose that $\mathcal{B}$ is a base for a filter $\mathscr{F}$. Let $U, V \in \mathcal{B}$. Since $\mathcal{B} \subset \mathscr{F}$, we have that $U \cap V \in \mathscr{F}$. Using that $\mathcal{B}$ is a base, there is $W \in \mathcal{B}$ such that $W \subset U \cap V$.

Now, suppose that for any $U, V \in \mathcal{B}$ there is $W \in \mathcal{B}$ such that $W \subset U \cap V$. Define

$$\mathscr{F} := \{U \subset X | U' \subset U, \text{ for some } U' \in \mathcal{B}\}$$

In order to see that $\mathscr{F}$ is a filter:

- Let $U \in \mathscr{F}$ and $V \subset X$ such that $U \subset V$. By definition of $\mathscr{F}$ there is $U' \in \mathcal{B}$ such that $U' \subset U \subset V$. Thus, $V \in \mathscr{F}$.
- Let $U, V \in \mathscr{F}$, then there are $U', V' \in \mathcal{B}$ such that $U' \subset U$ and $V' \subset V$. By hypothesis, there is $W \in \mathcal{B}$ such that $W \subset U' \cap V' \subset U \cap V$. Thus, $U \cap V \in \mathscr{F}$.

$\square$

Theorem 1.4. *Let (X, c) be a closure space and let $A \subset X$ be a subset. Then the neighborhood system $\mathcal{N}(A)$ of A is a filter on X whose intersection contains A.*

Proof. First note that $\mathcal{N}(A)$ is nonempty since $i(X) = X \supset A$. Now,

- Let $U, V \in \mathcal{N}(A)$. By hypothesis, $A \subset i(U)$ and $A \subset i(V)$. Using the property $(I3)$ of the interior operator we have that

  $$i(U \cap V) = i(U) \cap i(V)$$

  Thus, $A \subset i(U \cap V)$, and so $U \cap V \in \mathcal{N}(A)$.

- Let $U \in \mathcal{N}(A)$ and $V \subset X$ such that $U \subset V$. By hypothesis, $A \subset i\,(U)$. Note that

$$i\,(U) = i\,(U \cap V) = i\,(U) \cap i\,(V) \subset i\,(V)$$

  Thus, $A \subset i\,(V)$ and so $V \in \mathcal{N}(A)$.

$\square$

Definition 1.6. Consider the neighborhood system of a A in (X, c). A (sub-)base of this filter is called a *(sub-)base of the neighborhood system of* A *in* X. If $A = \{x\}$, for some $x \in X$, the term *local (sub-)base at* x will be used instead.

Observation 2. Consider a closure space (X, c) and a subset $A \subset X$. Using the definition for bases and subbases of filters, we obtain the following properties:

- A collection $\mathcal{V}$ of subsets of X is a base of the neighborhood system of A in X if and only if each $V \in \mathcal{V}$ is a neighborhood of A and every neighborhood of A contains a $V \in \mathcal{V}$.
- A collection $\mathcal{W}$ of subsets of X is a subbase of the neighborhood system of A in X if and only if all finite intersections of elements in $\mathcal{W}$ is a base of the neighborhood system of A.

Observation 3. Let $\mathcal{B}_x$ be a local base at x. Then the following are immediate from the definitions:

(B1) $\mathcal{B}_x \neq \emptyset$.

(B2) For each $U \in \mathcal{B}_x,\ x \in U$.

(B3) For each $U_1, U_2 \in \mathcal{B}_x$ there is $U \in \mathcal{B}_x$ such that $U \subset U_1 \cap U_2$.

We have seen that a closure can be induced by the interior. Similarly we have that we can define the closure with the neighborhoods at each point.

Theorem 1.5. *(Corrolary 14.B7[2]) Let (X, c) be a closure space, $A \subset X$ a subset, and consider a point $x \in X$. Then $x \in c\,(A)$ if and only if $A \cap U \neq \emptyset$, for each $U \in \mathcal{B}_x$, where $\mathcal{B}_x$ is a local base at x.*

Proof. Suppose there is $U \in \mathcal{B}_x$ such that $U \cap A = \emptyset$. Then $U \subset X \setminus A$, and so

$$i\,(U) \subset i\,(X \setminus A) = X \setminus c\,(A)$$

Using that $x \in i\,(U)$, it follows that $x \notin c\,(A)$.

Now suppose that $x \notin c\,(A)$. Using that $X \setminus c\,(A) = i\,(X \setminus A)$, we conclude that $X \setminus A$ is a neighborhood of x. Since $\mathcal{B}_x$ is a local base, there is $U \in \mathcal{B}_x$ such that $U \subset X \setminus A$, and so $U \cap A \subset (X \setminus A) \cap A = \emptyset$. $\square$

As in the case for the closure operator, we have a characterization of the interior operator using a local basis.

Theorem 1.6. *Let (X, c) be a closure space, $A \subset X$ a subset, and consider a point $x \in X$. Then $x \in i(A)$ if and only if there is $U \in \mathcal{B}_x$ such that $U \subset A$, where $\mathcal{B}_x$ is a local base at x.*

Proof. Suppose $x \in i(A)$, then $A \in \mathcal{N}_x$. Since $\mathcal{B}_x$ is a local base of the neighborhood system $\mathcal{N}_x$, there is $U \in \mathcal{B}_x$ such that $U \subset A$.

Now, suppose that there is $U \in \mathcal{B}_x$ such that $U \subset A$. It follows that then $i(U) \subset i(A)$. Since $U \in \mathcal{B}_x \subset \mathcal{N}_x$, we have that U is a neighborhood of x, i.e., $x \in i(U) \subset i(A)$. □

We've shown that for a closure space there is a special filter on each point called the neighborhood filter. The converse is also true, i.e., given a filter for each point we can obtain a closure such that these filters are the local neighborhood systems. Since a filter can be recover with a base for it, we can consider the base instead of the whole filter. This serves as a motivation for con the following theorem.

Theorem 1.7. *(Theorem 14B.10 [2]) Let X be a set and for each $x \in X$ let $\mathcal{B}_x$ be a collection of subsets of X satisfying the conditions (B1), (B2) and (B3) of Observation 3. Then there is an unique closure operator c for X such that, for each $x \in X$, $\mathcal{B}_x$ is a local base at x for (X, c).*

Proof. The Theorem 1.5 suggests us that we should define an operator $c : \mathscr{P}(X) \to \mathscr{P}(X)$ by

$$c(U) := \{x \in X |\ V \cap U \neq \emptyset,\ \forall V \in \mathcal{B}_x\}$$

We must prove that in fact c is a closure operator and that $\mathcal{B}_x$ are local bases at x in (X, c), for each $x \in X$.

- Proof of (C1)
  Note that for all $x \in X$ and $V \in \mathcal{B}_x$, $V \cap \emptyset = \emptyset$. Thus, $c(\emptyset) = \emptyset$.

- Proof of (C2)
  Let $U \subset X$. Using (B2), we have that, if $x \in U$ and $V \in \mathcal{B}_x$, then $x \in V$. Therefore, $x \in V \cap U$ and so $x \in c(U)$. Thus $U \subset c(U)$.

- Proof of (C3)
  Let $V_1, V_2 \subset X$. Suppose that $x \in c(V_1) \cup c(V_2)$. By definition, for each $V \in \mathcal{B}_x$, we have that $V_1 \cap V \neq \emptyset$ and $V_2 \cap V \neq \emptyset$. Thus, $(V_1 \cup V_2) \cap V \neq \emptyset$, and so $x \in c(V_1 \cup V_2)$.

  Now, suppose $x \notin c(V_1) \cup c(V_2)$. By definition there are $W_1, W_2 \in \mathcal{B}_x$ such that $V_1 \cap W_1 = \emptyset$ and $V_2 \cap W_2 = \emptyset$. Using (B3), there is $W \in \mathcal{B}_x$ such that $W \subset W_1 \cap W_2$. It follows that

$$(V_1 \cup V_2) \cap W = (V_1 \cap W) \cup (V_2 \cap W) \subset (V_1 \cap W_1) \cup (V_2 \cap W_2) = \emptyset$$

Thus, $x \notin c(V_1 \cup V_2)$.

This proves that c is in fact a closure for X.

Now we need to show that each $\mathcal{B}_x$ is a local base at x.

Let $x \in X$ and $U \in \mathcal{B}_x$. We know that $U \cap (X \setminus U) = \emptyset$ and so $x \notin c(X \setminus U)$. This means that

$$x \in X \setminus c(X \setminus U) = i(U)$$

Thus, U is a neighborhood of x.

Now, let W be a neighborhood of x. This means that $x \notin c(X \setminus W)$. By definition of c, there is $U \in \mathcal{B}_x$ such that

$$U \cap (X \setminus W) = \emptyset$$

Therefore, $U \subset W$.

In conclusion, for each $x \in X$, $\mathcal{B}_x$ is a local base at x for (X, c). □

The following is an immediate corollary, using the definition of the filter.

Corollary. *(Corollaries 14 B.11 [2])*

1. *For each $x \in X$, let $\mathcal{N}_x$ be a filter on X such that $x \in \cap\mathcal{N}_x$. Then there is an unique closure operator for X such that $\mathcal{N}_x$ is the neighborhood system at x in (X, c), $\forall x \in X$.*

2. *For each $x \in X$, let γ_x be a nonempty family of subset of X such that $x \in \cap\gamma_x$. Then there is an unique closure operator for X such that, for each $x \in X$, γ_x is a local subbase at x in (X, c).*

Proposition 1.8. *(Theorem 16 A.4 [2]) Let $f : (X, c_X) \to (Y, c_Y)$ be a map between closure spaces. Then, f is continuous if and only if, for each $x \in X$ and $V \in \mathcal{N}_{f(x)}$, we have that $f^{-1}(V) \in \mathcal{N}_x$, i.e., the inverse image of a neighborhood of $f(x)$ is a neighborhood of x.*

Proof. First fix $x \in X$. Suppose f is continuous. Using Proposition 1.2, if $V \in \mathcal{N}_{f(x)}$, i.e., $f(x) \in i_Y(V)$, then

$$\begin{aligned} x &\in f^{-1}(f(x)) \\ &\subset f^{-1}(i_Y(V)) \\ &\subset i_x\left(f^{-1}(V)\right) \end{aligned}$$

Thus, $f^{-1}(V) \in \mathcal{N}_x$.

Now, suppose that, for each $V \in \mathcal{N}_{f(x)}$, we have that $f^{-1}(V) \in \mathcal{N}_x$, and consider $U \subset X$ such that $f(x) \notin c_Y(f(U))$. It follows that

$$f(x) \in Y \setminus c_Y(f(U)) = i_Y(Y \setminus f(U))$$

and so, $Y \setminus f(U)$ is a neighborhood of $f(x)$. By hypothesis, $f^{-1}(Y \setminus f(U))$ is a neighborhood of x. Note that $f^{-1}(Y \setminus f(U)) \cap U = \emptyset$. It follows that $f(x) \notin c_Y(f(U))$ implies that $x \notin c_X(U)$. Thus, if $x \in c_X(U)$, then $f(x) \in c_Y(f(U))$. Since x was any element of X, we have that

$$f(c_X(U)) \subset c_Y(f(U)),$$

i.e, f is continuous. ☐

The tools we just described above will be helpful for the following constructions of closure spaces.

Definition 1.7. Let X be a set and two closure operators c_1, c_2 for X. If the identity map $\mathrm{Id}_X : (X, \mathrm{c}_1) \to (X, \mathrm{c}_2)$ is continuous, we say that c_2 is weaker (coarser) than c_1 and that c_1 is stronger (finner) than c_2. This means that for any $U \subset X$

$$c_1(U) = \mathrm{Id}_X(c_1(U)) \subset c_2(\mathrm{Id}_X(U)) = c_2(U).$$

We denote this relation by $c_2 \leq c_1$.

Now consider two closure spaces $(X, \mathrm{c}_X), (Y, \mathrm{c}_Y)$. We would like to construct a closure on the corresponding Cartesian product $X \times Y$. For each $(x, y) \in X \times Y$, define the collection of the sets

$$\gamma_{(x,y)} = {\pi_x}^{-1}(\mathcal{N}_x) \cup {\pi_y}^{-1}(\mathcal{N}_y) = \{{\pi_x}^{-1}(U) \,|U \in \mathcal{N}_x\} \cup \{{\pi_y}^{-1}(V) \,|V \in \mathcal{N}_y\}$$

where π_x, π_y are the respective projections. By Corollary 1, this collection induces a closure $c_{X,Y}$ such that each $\gamma_{(x,y)}$ is a local subbase at (x, y). Thus, the finite intersections of its elements are a local base, i.e.,

$$\mathcal{B}_{(x,y)} = \{U \times V |\ U \in \mathcal{N}_x,\ V \in \mathcal{N}_y\}$$

is a local base at (x, y).

Definition 1.8. Given two closure spaces (X, c_X) and (Y, c_Y), we define a closure operator $c_{X,Y}$ for $X \times Y$ as above. We say this closure is the *product closure for* $X \times Y$.

Lemma 1.9. *The natural projections*

$$\pi_x : (X \times Y, c_{X,Y}) \to (X, c_X) \quad \textit{and} \quad \pi_y : (X \times Y, c_{X,Y}) \to (Y, c_Y)$$

are continuous.

Proof. Let $(x, y) \in X \times Y$, $U \in \mathcal{N}_x$, and $V \in \mathcal{N}_y$. Then ${\pi_x}^{-1}(U) = U \times Y \in \mathcal{N}_{(x,y)}$ and ${\pi_y}^{-1}(V) = X \times V \in \mathcal{N}_{(x,y)}$. Using Proposition 1.8, we have that π_x and π_y are continuous. ☐

Proposition 1.10. *Let $(X \times Y, c_{X,Y})$ be the product of two closure spaces (X, c_X), (Y, c_Y). Then for all $A \subset X$ and $B \subset Y$:*

- $c_{X,Y}(A \times B) = c_X(A) \times c_Y(B)$.
- $i_{X,Y}(A \times B) = i_X(A) \times i_Y(B)$.

Proof. Given $A \subset X$ and $B \subset Y$. Consider $(x, y) \in c_{X,Y}(A \times B)$. Remember that $\mathcal{B}_{(x,y)} := \{U \times V | U \in \mathcal{N}_x,\ V \in \mathcal{N}_y\}$ is a local base at (x, y) in the product closure, where $\mathcal{N}_x$ and $\mathcal{N}_y$ are the neighborhood systems at x and y respectively. Using Theorem 1.5, we have that for any $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$

$$(A \times B) \cap (U \times V) \neq \emptyset$$

It follows that

$$A \cap U \neq \emptyset \quad \text{and} \quad B \cap V \neq \emptyset$$

This means that $x \in c_X(A)$ and $y \in c_Y(B)$, i.e.,

$$(x, y) \in c_X(A) \times c_Y(B)$$

Thus, $c_{X,Y}(A \times B) \subset c_X(A) \times c_Y(B)$. Similarly, we have that $c_X(A) \times c_Y(B) \subset c_{X,Y}(A \times B)$.

Now suppose $(x, y) \in i_{X,Y}(A \times B)$. Then there is $U \times V \in \mathcal{B}_{(x,y)}$ such that $U \times V \subset A \times B$, with $U \in \mathcal{N}_x$ and $V \in \mathcal{N}_y$. It follows that $U \subset A$ and $V \subset B$, and so

$$i_{X,Y}(A \times B) \subset i_X(A) \times i_Y(B)$$

Similarly we have that $i_X(A) \times i_Y(B) \subset i_{X,Y}(A \times B)$. $\square$

Proposition 1.11. *Given two continuous functions between closure spaces $f : (Z, c_z) \to (X, c_X)$ and $g : (Z, c_z) \to (Y, c_Y)$ there is a unique continuous function $(f, g) : (Z, c_z) \to (X \times Y, c_{X,Y})$ such that $\pi_x(f, g) = f$ and $\pi_y(f, g) = g$, i.e., the following diagram commutes*

$$\begin{tikzcd}
 & Z \arrow[dl, "f"'] \arrow[d, dashed, "{(f,g)}"] \arrow[dr, "g"] & \\
X & X \times Y \arrow[l, "\pi_x"] \arrow[r, "\pi_y"'] & Y
\end{tikzcd}$$

Proof. We know, from the category of sets, there is a unique map (f, g) such that $\pi_x(f, g) = f$ and $\pi_y(f, g) = g$ defined as

$$(f, g)(z) = (f(z), g(z))$$

So we need to prove that (f, g) is continuous.

Let $z \in Z$, $U_z \in \mathcal{N}_{f(z)}$ and $V_z \in \mathcal{N}_{g(z)}$. Then

$$f(z) \in i_X(U_z) \quad \text{and} \quad g(z) \in i_X(V_z)$$

Using Proposition 1.8, we have that $f^{-1}(U_z), g^{-1}(V_z) \in \mathcal{N}_z$. Note that

$$\begin{aligned}(f,g)^{-1}(U_z \times V_z) &:= \{w \in Z |\ (f,g)(w) \in U_z \times V_z\} \\ &= \{w \in Z |\ f(w) \in U_z,\ g(w) \in V_z\} \\ &= f^{-1}(U_z) \cap g^{-1}(V_z) \in \mathcal{N}_z\end{aligned}$$

This means that the inverse image of the local base $\mathcal{B}_{(f(z),g(z))}$ is a subset of the neighborhoods of z. Remember that $\mathcal{N}_z$ is a filter and that the inverse image of the union of elements of $\mathcal{B}_{(f(z),g(z))}$ is the union of the inverse images of elements of $\mathcal{B}_{(f(z),g(z))}$. Thus, the inverse image of a neighborhood of $(f(z), g(z))$ is a neighborhood of z. Using proposition 1.8, we conclude that (f, g) is a continuous function. □

Corollary. *Given two closure spaces $(X, c_X), (Y, c_Y)$, the product closure $c_{X,Y}$ is the coarsest closure operator for $X \times Y$ such that the natural projections*

$$\pi_x : (X \times Y, c_{X,Y}) \to (X, c_X) \quad \textit{and} \quad \pi_y : (X \times Y, c_{X,Y}) \to (Y, c_Y)$$

are continuous.

Proof. Let c be a closure for $X \times Y$ such that the projections π_x and π_y are continuous. Using Proposition 1.11, there is a unique continuous map between $(X \times Y, \mathrm{c})$ and $(X \times Y, \mathrm{c}_{X,Y})$ that commutes with the natural projections.

$$\begin{tikzcd} & (X \times Y, \mathrm{c}) \arrow[dl, "\pi_x"'] \arrow[d, dashed, "(\pi_x,\pi_y)"] \arrow[dr, "\pi_y"] & \\ (X, \mathrm{c}_X) & (X \times Y, \mathrm{c}_{X,Y}) \arrow[l, "\pi_x"] \arrow[r, "\pi_y"'] & (Y, \mathrm{c}_Y) \end{tikzcd}$$

Since the identity is the only map that makes the diagram commutative, we have that

$$\mathrm{Id}_{X \times Y} : (X \times Y, \mathrm{c}) \to (X \times Y, \mathrm{c}_{X,Y})$$

is continuous, and so $c_{X,Y}$ is coarser than c.

Since c was arbitrary and $c_{X,Y}$ is itself a closure for $X \times Y$ such that the natural projections π_x and π_y are continuous, we have that $c_{X,Y}$ is the coarsest closure operator for $X \times Y$ such that the natural projections π_x and π_y are continuous. □

Now consider a closure space (X, c). For $A \subset X$ we would like to define a closure for A

compatible with the closure for X. So, for each $a \in A$ consider the collection

$$\mathcal{M}_a := \{U \cap A |\, U \in \mathcal{N}_a\}$$

where $\mathcal{N}_a$ is the neighborhood system at a. We will show that $\mathcal{M}_a$ satisfies (B1), (B2), and (B3):

- Proof of (B1)
  Since $X \in \mathcal{N}_a$, we have that $A \in \mathcal{M}_a$, and so $\mathcal{M}_a \neq \emptyset$.

- Proof of (B2)
  For each $V \in \mathcal{M}_a$ there is $U \in \mathcal{N}_a$, a neighborhood of a, such that $V = U \cap A$. Since $a \in U$, we have that $a \in U \cap A = V$.

- Proof of (B3)
  If $V_1, V_2 \in \mathcal{M}_a$, there are $U_1, U_2 \in \mathcal{N}_a$ such that $V_\alpha = U_\alpha \cap A$, for $\alpha = 1, 2$. Since $U_1 \cap U_2 \in \mathcal{N}_a$, we have that
  $$V_1 \cap V_2 = (U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A \in \mathcal{M}_a$$

Using Theorem 1.7, there is a unique closure c_A for A such that each $\mathcal{M}_a$ is a local base at a.

Definition 1.9. Let (X, c_X) be a closure space and a subset $A \subset X$. Define a closure operator c_A for A as above. We say c_A is the *subspace closure for A*.

Proposition 1.12. *Let (X, c_X) be a closure space. Consider $A \subset X$ and the natural inclusion $\iota : A \to X$. Then the subspace closure and interior operators defined on A satisfies:*

- $c_A(U) = c_X(U) \cap A$, *for all* $U \subset A$.

- $i_A(U) = i_X(U \cup (X \setminus A)) \cap A$, *for all* $U \subset A$.

Proof.
- Let $U \subset A$. Consider $a \in c_A(U) \subset A$. Given $V' \in \mathcal{N}_a$ a neighborhood of a in X. If $\mathcal{M}_a$ is the local base as in the definition of the subspace closure, then $V := V' \cap A \in \mathcal{M}_a$ and
  $$\emptyset \neq V \cap U = (V' \cap A) \cap U = V' \cap (A \cap U) = V' \cap U$$
  Thus, $a \in c_X(U) \cap A$, and so
  $$c_A(U) \subset c_X(U) \cap A$$
  Now, let $a \in c_X(U) \cap A$. Given $V \in \mathcal{M}_a$ there is $V' \in \mathcal{N}_a$ such that $V = A \cap V'$. It follows that
  $$\emptyset \neq U \cap V' = (U \cap A) \cap V' = U \cap (A \cap V') = U \cap V$$
  Thus, $a \in c_A(U)$, and so $c_X(U) \cap A \subset c_A(U)$.

- From the definition of the interior, we have that $A \setminus i_A(U) = c_A(A \setminus U)$. Note that

$$\begin{aligned} X \setminus i_X(U \cup (X \setminus A)) &= c_X(X \setminus (U \cup (X \setminus A))) \\ &= c_X((X \setminus U) \cap (X \setminus (X \setminus A))) \\ &= c_X((X \setminus U) \cap A)) \\ &= c_X(A \setminus U) \end{aligned}$$

Then

$$\begin{aligned} A \setminus i_X(U \cup (X \setminus U)) &= A \cap X \setminus i_X(U \cup (X \setminus U)) \\ &= A \cap c_X(A \setminus U) \\ &= c_A(A \setminus U) \\ &= A \setminus i_A(U) \end{aligned}$$

Thus, $i_A(U) = i_X(U \cup (X \setminus U)) \cap A$.

□

Corollary. *The natural inclusion $\iota : (A, c_A) \to (X, c_X)$ is continuous.*

Proof. For any $U \subset A$

$$\iota(c_A(U)) = c_A(U) = c_X(U) \cap A \subset c_X(U) = c_X(\iota(U))$$

and so the inclusion ι is continuous. □

Proposition 1.13. *Let (X, c_X) be a closure space and a subset $A \subset X$. Given a function between closure spaces $f : (Z, c_z) \to (A, c_A)$, if c_A is the subspace closure for A and $\iota : (A, c_A) \to (X, c_X)$ is the natural inclusion. Then f is continuous if and only if ιf is continuous.*

$$\begin{tikzcd} & (X, c_X) \\ (Z, c_z) \arrow[ur, "\iota f"] \arrow[r, "f"'] & (A, c_A) \arrow[u, "\iota"'] \end{tikzcd}$$

Proof. We have shown that ι is continuous. Thus, if f is continuous, then ιf is continuous.

Suppose that ιf is continuous. Let $z \in Z$ and $V \in \mathcal{M}_{f(z)}$, there is $V' \in \mathcal{N}_{f(z)} = \mathcal{N}_{(\iota f)(z)}$ such that $V = V' \cap A$. Using Proposition 1.8, we have that $f^{-1}(V')$ is a neighborhood of z. Note that

$$f^{-1}(V') = f^{-1}(V') \cap f^{-1}(A) = f^{-1}(V' \cap A) = f^{-1}(V)$$

Therefore, f is continuous. $\square$

Corollary. *The subspace closure c_A is the coarsest closure operator such that the natural inclusion*

$$\iota : (A, c_A) \to (X, c_X)$$

is continuous.

Proof. If c is a closure for A such that $\iota : A \to X$ is continuous. Using the following commutative diagram

$$\begin{tikzcd} & (X, \mathrm{c}_X) \\ (A, \mathrm{c}) \arrow[ur, "\iota"] \arrow[r, "\mathrm{Id}_A"'] & (A, \mathrm{c}_A) \arrow[u, "\iota"'] \end{tikzcd}$$

and Proposition 1.13, we have that $\mathrm{Id}_A : (A, \mathrm{c}) \to (A, \mathrm{c}_A)$ is continuous, i.e., c_A is coarser than c. Since c is arbitrary, we conclude that c_A is the coarsest closure that makes the natural inclusion continuous. $\square$

Chapter 2

Čech (co)homology

In this chapter we will define the Čech (co)homology for closure spaces.

2.1 Interior Covers

In order to construct the Čech (co)homology for closure spaces, first we need to discus what covers means in the context of closure spaces.

Definition 2.1. Given a closure space (X, c), a collection of subsets $\mathscr{U} \subset \mathscr{P}(X)$ is *an interior cover of* X if

$$X = \bigcup_{U \in \mathscr{U}} i(U)$$

We denote by $\Gamma(X)$ to the collection of all interior covers of X. If A is a subspace of X, and $\mathscr{U}_A \subset \mathscr{U}$ is such that

$$A \subset \bigcup_{U \in \mathscr{U}_A} i(U),$$

then we say that the pair $(\mathscr{U}, \mathscr{U}_A)$ is an interior cover of the pair (X, A). We denote by $\Gamma(X, A)$ to the collection of all interior covers of the pair (X, A)

Definition 2.2. Let $\mathscr{U}, \mathscr{V} \in \mathscr{P}(X)$, two collections of subsets of X. We say that $\mathscr{V}$ is a *refinement* of $\mathscr{U}$ if every set $V \in \mathscr{V}$ is contained in some $U \in \mathscr{U}$. We denote this relationship by $\mathscr{U} < \mathscr{V}$.

Remark. We have that $\Gamma(X)$ is a *partially ordered set* with the "refinement" relation describe before. Also note that this partial order can be extended to the interior covers of the pair (X, A). Let $(\mathscr{U}, \mathscr{U}_A), (\mathscr{V}, \mathscr{V}_A) \in \Gamma(X, A)$, then we say that $(\mathscr{V}, \mathscr{V}_A)$ is a refinement of $(\mathscr{U}, \mathscr{U}_A)$ if $\mathscr{U} < \mathscr{V}$ and $\mathscr{U}_A < \mathscr{V}_A$. With this relation, we have that in fact $\Gamma(X, A)$ is a partial order.

Example 2.1. Let $G = (V, E)$ be an undirected graph without loops, i.e., $\{x, x\} \notin E$, for each $x \in V$. Then we can define a closure operator over V, using E. We start by defining the closure operator on each point $x \in V$ as

$$c(x) = \{y \in V : \{x, y\} \in E, \text{ or } y = x\}$$

and then extending it over unions, i.e.,

$$c(A) = \bigcup_{a \in A} c(a).$$

Observation 4. From the definition and the fact that G is undirected, we have that for any $x, y \in V$

$$x \in c(y) \Leftrightarrow y \in c(x) \tag{2.1}$$

Furthermore using the definition on the interior and closure operators, if $U \subset V$, then we have that

$$\begin{aligned} i(U) &= V \setminus c(V \setminus U) \\ &= V \setminus \left(\bigcup_{y \in V \setminus U} c(y) \right) \\ &= \bigcap_{y \in E \setminus U} E \setminus c(y) \end{aligned} \tag{2.2}$$

In this particular example, the following is true for any point $x \in V$ and subset $U \subset V$:

$$x \in i(U) \Leftrightarrow c(x) \subset U$$

This also shows that $x \in i(c(x))$.

($\Rightarrow$) Suppose that

$$x \in i(U) = \bigcap_{y \in V \setminus U} V \setminus c(y)$$

Then, for each $y \in V \setminus U$ we have that $x \in V \setminus c(y)$. Using (2.1), we have that $x \in V \setminus c(y)$ if and only if $y \in V \setminus c(x)$. Thus, for each $y \in V \setminus U$ we have that $y \in V \setminus c(x)$, i.e., $V \setminus U \subset V \setminus c(x)$. Therefore, $c(x) \subset U$.

($\Leftarrow$) Now suppose that $c(x) \subset U$. Then we have that $V \setminus U \subset V \setminus c(x)$, i.e., for each $y \in V \setminus U$ we have that $y \in V \setminus c(x)$. Finally, using (2.1) and (2.2), we conclude that

$$x \in \bigcap_{y \in V \setminus U} V \setminus c(y) = i(U)$$

Now let $\mathscr{U}$ be any interior cover of V. Define

$$\mathscr{V} := \{c(x) \,|x \in X\}$$

Since $\mathscr{U}$ is an interior cover, we have that for each $x \in E$ there is $U \in \mathscr{U}$ such that $x \in i(U)$. Using the previous result, we have that $c(x) \subset U$, and so $\mathscr{V}$ is a refinement of $\mathscr{U}$.

Note that $\mathscr{V}$ is itself an interior cover. Since $\mathscr{V}$ is also a refinement for all interior covers, we conclude that $\mathscr{V}$ is the supremum over all interior covers. This will be useful since we are going to use inverse (and direct) limits in order to define the Čech (co)homology.

Definition 2.3. Given a closure space (X, c) and an interior cover $\mathscr{U}$ of X, we define the *nerve of the cover* $\mathscr{U}$ to be the simplicial complex $K_{\mathscr{U}}$ whose vertices are the elements of $\mathscr{U}$, and where the set of n simplices is

$$\left\{\{U_0, \ldots, U_n\}| \bigcap_{i=0}^{n} U_i \neq \emptyset\right\}.$$

Definition 2.4. Given a pair (X, A), and a cover $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$. Define the *subcomplex of $K_{\mathscr{U}}$ associated with the subspace A* to be the subcomplex $L_{\mathscr{U}_A}$ of $K_{\mathscr{U}}$ such that a simplex $\{U_0, \ldots, U_n\}$ of $K_{\mathscr{U}}$ is also a simplex of $L_{\mathscr{U}_A}$ if and only if each $U_j \in \mathscr{U}_A$, and $U_0 \cap \ldots \cap U_n \cap A \neq \emptyset$.

Remark. This construction associates to each pair (X, A) of closure spaces, such that $A \subset X$, and cover $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$ a pair of simplicial complexes that we are going to use in order to define the Čech homology (and cohomology) of the space.

Definition 2.5. Given a pair of simplicial complexes (K, L), we denote $H_n(K, L)$ and $H^n(K, L)$ to be the n^{th} homology and cohomology groups of the pair (K, L).

Definition 2.6. Given a closure space pair (X, A) along with a interior cover $(\mathscr{U}, \mathscr{U}_A)$, we define

$$H_n(X, A; \mathscr{U}, \mathscr{U}_A) := H_n(K_{\mathscr{U}}, L_{\mathscr{U}_A}), \quad \text{and} \quad H^n(X, A; \mathscr{U}, \mathscr{U}_A) := H^n(K_{\mathscr{U}}, L_{\mathscr{U}_A}),$$

the n^{th} homology and cohomology groups of the pair (X, A) relative to the cover $(\mathscr{U}, \mathscr{U}_A)$.

Definition 2.7. A simplicial complex K is called acyclic if it has the same (co)homology groups as the single point space.

2.2 Homomorphisms on refinements

Definition 2.8. Let $f, g : (K_1, L_1) \to (K_2, L_2)$ be simplicial maps between simplicial pairs. We say that they are *contiguous* if for every simplex S in K_1 there is a simplex S' in K_2 containing both $f(S) \cup g(S)$. Furthermore, if S is in L_1, then S' is in L_2.

Definition 2.9. Let $C : K \to K'$ be a map (which may not be a simplicial map) between simplicial complexes. We say that C is a *carrier function* if, for each simplex S of K, $C(S)$ is a nonempty subcomplex of K' and if, for every face S' of S, $C(S')$ is a subcomplex of $C(S)$.

If, for every simplex S of K, the complex $C(S)$ is acyclic, we say that C is an *acyclic carrier*.

Definition 2.10. If $f : K \to K'$ is a simplicial map such that for any $S' \subset S$ we have that $f(S') \subset C(S)$, then C is called a *carrier of f*.

The following result can be found in [10], but the proof will be omitted since the theory necessary is outside of the scope of this Thesis.

Theorem 2.1 (5.8, Chaper VI [10]). *Let $f, g : K_1 \to K_2$ be simplicial maps with an acyclic carrier C. Then $f_* = g_*$ and $f^* = g^*$.*

The following is going to be an essential result that will be used constantly after and is a direct result of the previous Theorem.

Lemma 2.2 ([10]). *Let $f, g : (K_1, L_1) \to (K_2, L_2)$ be simplicial maps that are contiguous. Then f and g are homotopic, and so they induce the same homomorphisms on simplicial homology groups.*

Proof. For each simplex S of K_1 define $C(S)$ as the least simplex of K_2 that contains both $f(S)$ and $g(S)$. Since each simplex is acyclic, we have that C is an acyclic carrier. Thus, using Theorem 2.1 we conclude that in fact

$$f_* = g_* \quad \text{and} \quad f^* = g^*.$$

□

Proposition 2.3. *Give a closure space pair (X, A) and two interior covers $(\mathscr{U}, \mathscr{U}_A), (\mathscr{V}, \mathscr{V}_A) \in \Gamma(X, A)$. If $(\mathscr{U}, \mathscr{U}_A) < (\mathscr{V}, \mathscr{V}_A)$, then there exists a simplicial map $\pi^1_{\mathscr{U}\mathscr{V}} : (K_{\mathscr{V}}, L_{\mathscr{V}_A}) \to (K_{\mathscr{U}}, L_{\mathscr{U}_A})$, defined up to contiguity.*

Proof. Let V be a vertex in $K_{\mathscr{V}}$, i.e., $V \in \mathscr{V}$. Since $\mathscr{U} < \mathscr{V}$, there is some set $U \in \mathscr{U}$ such that $V \subset U$. So we can define $\pi^1_{\mathscr{U}\mathscr{V}}$ on the vertices of $K_{\mathscr{V}}$, choosing such a U for each $V \in \mathscr{V}$.

Now we need to show that $\pi^1_{\mathscr{U}\mathscr{V}}$ can be extended to a simplicial map. Take vertices $V_0, \ldots, V_n$ of a simplex of $K_{\mathscr{V}}$, and let $U_0, \ldots, U_n$ be the respective images under $\pi^1_{\mathscr{U}\mathscr{V}}$. Note that

$$\emptyset \neq V_0 \cap \ldots \cap V_n \subset U_0 \cap \ldots \cap U_n$$

Therefore $U_0, \ldots, U_n$ are vertices of a simplex of $K_{\mathscr{U}}$. Now consider $L_{\mathscr{U}_A},\ L_{\mathscr{V}_A}$ the subcomplexes of $K_{\mathscr{U}},\ K_{\mathscr{V}}$ associated with A, respectively. If $V_0, \ldots, V_n$ are vertices of a simplex of $L_{\mathscr{V}_A}$, it means that

$$\emptyset \neq A \cap V_0 \cap \ldots \cap V_n \subset A \cap U_0 \cap \ldots \cap U_n$$

so we have that $U_0, \dots, U_n$ are vertices of a simplex in $L_{\mathscr{U}_A}$. Thus $\pi^1_{\mathscr{U}\mathscr{V}}$ can be extended as desired.

Now, for each $V \in \mathscr{V}$, we define another map by making a second choice $W \in \mathscr{U}$ such that $V \subset W$. Let $\pi^2_{\mathscr{U}\mathscr{V}}$ be this map sending V to W. Let $V_0, \dots, V_n$ vertices of a simplex of $K_{\mathscr{V}}$ and let $\pi^1_{\mathscr{U}\mathscr{V}}(V_j) = U_j,\ \pi^2_{\mathscr{U}\mathscr{V}}(V_j) = W_j$, with $j = 1, \dots, n$. Note that each $V_j \subset U_j,\ V_j \subset W_j$, so

$$\emptyset \neq V_0 \cap \dots \cap V_n \subset U_1 \cap \dots \cap U_n \cap W_0 \dots \cap W_n$$

and thus follows that $\pi^1_{\mathscr{U}\mathscr{V}}, \pi^2_{\mathscr{U}\mathscr{V}}$ are contiguous and each maps the pair $(K_{\mathscr{V}},\ L_{\mathscr{V}_A})$ to $(K_{\mathscr{U}},\ L_{\mathscr{U}_A})$. □

Corollary. *The simplicial maps $\pi^1_{\mathscr{U}\mathscr{V}}$ and $\pi^2_{\mathscr{U}\mathscr{V}}$ induce the same homomorphism on homology*

$$\pi_{\mathscr{U}\mathscr{V}*} : H_n(X, A; \mathscr{V}, \mathscr{V}_A) \to H_n(X, A; \mathscr{U}, \mathscr{U}_A)$$

and the same homomorphism on cohomology

$$\pi^*_{\mathscr{V}\mathscr{U}} : H^n(X, A; \mathscr{U}, \mathscr{U}_A) \to H^n(X, A; \mathscr{V}, \mathscr{V}_A)$$

We call them the homomorphisms associated with the pair of covers $(\mathscr{U}, \mathscr{U}_A) < (\mathscr{V}, \mathscr{V}_A)$.

Note. We only write on the subindex one of the elements of the pair for clarity on the notation.

Theorem 2.4. *Let $\mathscr{U}, \mathscr{V}, \mathscr{W} \in \Gamma(X)$ such that $\mathscr{U} < \mathscr{V} < \mathscr{W}$, then*

$$\pi_{\mathscr{U}\mathscr{V}*}\pi_{\mathscr{V}\mathscr{W}*} = \pi_{\mathscr{U}\mathscr{W}*} \quad \text{and} \quad \pi^*_{\mathscr{W}\mathscr{V}}\pi^*_{\mathscr{V}\mathscr{U}} = \pi_{\mathscr{W}\mathscr{U}*}$$

Proof. Take $W \in \mathscr{W}$ to be a vertex of $K_{\mathscr{W}}$. Since $\mathscr{V} < \mathscr{W}$, there exists a $V \in \mathscr{V}$ such that $W \subset V$. Also $\mathscr{U} < \mathscr{V}$ implies there's $U \in \mathscr{U}$ such that $V \subset U$. Thus, we may define $\pi^1_{\mathscr{V}\mathscr{W}}(W) := V$, $\pi^1_{\mathscr{U}\mathscr{V}}(V) := U$, and $\pi^1_{\mathscr{U}\mathscr{W}}(W) := U$. If this is done for each vertex of $K_{\mathscr{W}}$, then we have that

$$\pi^1_{\mathscr{U}\mathscr{V}}\pi^1_{\mathscr{V}\mathscr{W}} = \pi^1_{\mathscr{U}\mathscr{W}}$$

is satisfy in the vertices and so, when extended by linearity, it will be satisfied on all $K_{\mathscr{W}}$. Then the induced homomorphisms on (co)homology are equal. □

Corollary. *For any $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$, the homomorphisms $\pi_{\mathscr{U}\mathscr{U}*}$ and $\pi^*_{\mathscr{U}\mathscr{U}}$ are the respective identity maps.*

2.3 Inverse Limits

The definition of limit (direct or inverse) can be applied in a more general context using Category Theory. Nevertheless, we will be taking a more elementary approach using directed sets and Abelian groups, and so we'll just be calling them groups.

We will follow the order from [12], but many of the proofs and some of the conclusions will differ to siut the context of this Thesis, and with a more emphasis to the universal properties of the inverse limit.

Definition 2.11. A *directed set* is a partially ordered set $(D, <)$ with the additional condition that for each pair of elements $\alpha,\ \beta \in D$ there is a element $\gamma \in D$ such that $\alpha, \beta < \gamma$. We will denote $(D, <)$ by D when the context is clear.

If A and D are directed sets, a map $f : A \to D$ is an *order-preserving function from A to D* if, for all $a, b \in A$ such that $a <_A b$, then $f(a) <_D f(b)$.

A directed set $(D', <_{D'})$ is a *subset* of $(D, <_D)$, denoted by $D' \subset D$, if $D' \subset D$ as a set, and the natural inclusion is an order-preserving map.

A subset D' is *cofinal* in D if, for each $a \in D$, there is $b \in D'$ such that $a < b$.

Definition 2.12. An *inverse system of groups* is a set of groups G_α, indexed by a directed set A such that for all $\alpha, \beta \in A$ with $\alpha < \beta$ there's a homomorphism $\pi_{\alpha\beta} : G_\beta \to G_\alpha$. These homomorphisms satisfy the conditions

1. $\pi_{\alpha\alpha} = \mathrm{Id}_{G_\alpha}$ for all $\alpha \in A$.

2. $\pi_{\alpha\beta}\pi_{\beta\gamma} = \pi_{\alpha\gamma}$, whenever $\alpha < \beta < \gamma$.

We will denote an inverse system of groups by $\{G_\alpha, \pi_{\alpha\beta}, A\}$. When the context allows it, this will abbreviated by $\{G_\alpha, \pi_{\alpha\beta}\}$ or simply $\{G_\alpha\}$.

Let $\{G_\alpha\}$ be an inverse system of groups. An element of the Cartesian product $g \in \prod\limits_{\alpha\in A} G_\alpha$ is specified by it's value in each coordinate, i.e., $g = (g_\alpha)$, with $g_\alpha \in G_\alpha$. Consider the subset

$$G := \left\{ (g_\alpha) \in \prod_{\alpha\in A} G_\alpha \;\middle|\; g_\alpha = \pi_{\alpha\beta}(g_\beta),\ \text{whenever } \alpha < \beta \right\} \subset \prod_{\alpha\in A} G_\alpha$$

Let $g, h \in G$, with $g = (g_\alpha)$ and $h = (h_\alpha)$. If $\alpha < \beta$, then we have that

$$g_\alpha - h_\alpha = \pi_{\alpha\beta}(g_\beta) - \pi_{\alpha\beta}(h_\beta) = \pi_{\alpha\beta}(g_\beta - h_\beta)$$

and so $g - h \in G$. Thus, G is a subgroup of $\prod\limits_{\alpha\in A} G_\alpha$.

Definition 2.13. The group G defined above is called *the inverse limit of the system* $\{G_\alpha\}$. We will denote it by

$$G = \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\}.$$

If there's no confusion, we will abbreviated this to $G = \varprojlim_{A}\{G_\alpha\}$ or simply $G = \varprojlim\{G_\alpha\}$.

Example 2.2. If A consists of one element α, then $\varprojlim\{G_\alpha\} = G_\alpha$.

Example 2.3. If A is an index set with a maximum element, i.e., there is $\beta \in A$ such that for any $\alpha \in A$, $\alpha < \beta$. Then $\varprojlim_{A}\{G_\alpha\} = G_\beta$.

Definition 2.14. For each $\beta \in A$, there are natural homomorphisms

$$\pi_\beta : \varprojlim\{G_\alpha\} \to G_\beta$$

corresponding to the composite $\varprojlim\{G_\alpha\} \xrightarrow{\iota} \prod_{\alpha\in A} G_\alpha \xrightarrow{\pi} G_\beta$ of the natural inclusion ι follow by the natural projection π. We say that π_β is *the projection of* $\varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\}$ *into* G_β.

Remark. Let $g = (g_\alpha) \in G$, and $\alpha < \beta$. Then, by construction,

$$\pi_\alpha(g) = g_\alpha = \pi_{\alpha\beta}(g_\beta) = \pi_{\alpha\beta}(\pi_\beta(g))$$

This means $\pi_\alpha = \pi_{\alpha\beta}\pi_\beta$, whenever $\alpha < \beta$.

Theorem 2.5 (Universal Property of inverse limits)**.** *Consider the inverse system of groups* $\{G_\alpha, \pi_{\alpha\beta}, A\}$. *Given a group* H *and homomorphisms* $\{f_\alpha : H \to G_\alpha\}_{\alpha\in A}$ *such that*

$$f_\alpha = \pi_{\alpha\beta} f_\beta$$

whenever $\alpha < \beta$*, there exits a unique homomorphism* $f : H \to \varprojlim\{G_\alpha\}$*, such that*

$$\pi_\alpha f = f_\alpha, \tag{2.3}$$

i.e., the following diagram commutes

$$\begin{tikzcd} & \varprojlim\{G_\alpha\} \arrow[d, "\pi_\alpha"] \\ H \arrow[ur, "f"] \arrow[r, "f_\alpha"'] & G_\alpha \end{tikzcd}$$

Proof. Let $h \in H$, define $g_\alpha := f_\alpha(h)$. Now, take $g = (g_\alpha) \in \prod_{\alpha\in A} G_\alpha$. Note that, by hypothesis,

$$g_\alpha = f_\alpha(h) = \pi_{\alpha\beta}(f_\beta(h)) = \pi_{\alpha\beta}(g_\beta).$$

Thus, $g \in \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\}$, and we define $f(h) := g = (g_\alpha)$ as constructed above.

Now we'll show that f is a homomorphism. Take $h_1, h_2 \in H$, then

$$\begin{aligned} f(h_1 + h_2) &= (f_\alpha(h_1 + h_2))_{\alpha \in A} \\ &= (f_\alpha(h_1) + f_\alpha(h_2))_{\alpha \in A} \\ &= (f_\alpha(h_1))_{\alpha \in A} + (f_\alpha(h_2))_{\alpha \in A} \\ &= f(h_1) + f(h_2). \end{aligned}$$

Thus, f is an homomorphism.

Finally we'll show that f is unique. Suppose $f' : H \to \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\}$ satisfies the condition (2.3). Given $h \in H$, we define $g := (f - f')(h)$. Note that the coordinates of g are

$$\begin{aligned} g_\alpha &= \pi_\alpha(g) \\ &= \pi_\alpha((f - f')(h)) \\ &= \pi_\alpha(f(h) - f'(h)) \\ &= \pi_\alpha(f(h)) - \pi_\alpha(f'(h)) \\ &= f_\alpha(h) - f_\alpha(h) \\ &= 0. \end{aligned}$$

and so $g = 0$. It follows that $f = f'$. $\square$

Now consider $B \subset A$, with B a directed set. Take all the groups G_β, with $\beta \in B$. Note that the relations between the homomorphisms $\pi_{\beta\gamma}$ remain even with the restricted indexes. This allows us to consider a new inverse system $\{G_\beta, \pi_{\beta\gamma}, B\}$. Thus we have two inverse limits $\varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\}$ and $\varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\}$. Since there are homomorphisms $\{\pi_\beta : \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\} \to G_\beta\}_{\beta \in B}$ such that for any $\beta, \gamma \in B$, with $\beta < \gamma$, we have that

$$\pi_\beta = \pi_{\beta,\gamma}\pi_\gamma,$$

i.e., the following diagram commutes

$$\begin{tikzcd} \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\} \arrow[r, "\pi_\gamma"] \arrow[dr, "\pi_\beta"'] & G_\gamma \arrow[d, "\pi_{\beta\gamma}"] \\ & G_\beta \end{tikzcd}$$

If π'_β is the natural projection from the inverse limit $\varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\}$ into G_β, then by Theorem 2.5

there's a unique homomorphism $\pi_{BA} : \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\} \to \varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\}$, such that

$$\pi'_\beta \pi_{BA} = \pi_\beta,$$

i.e., the following diagram commutes

$$\begin{tikzcd}
 & \varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\} \arrow[d, "\pi'_\beta"] \\
\varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\} \arrow[ur, "\pi_{BA}"] \arrow[r, "\pi_\beta"'] & G_\gamma
\end{tikzcd}$$

Definition 2.15. The homorphism π_{BA} is called the *projection map of* $\varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\}$ *into* $\varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\}$.

Remark. If $C \subset B \subset A$ are directed set, the uniqueness of the projection maps implies

$$\pi_{CA} = \pi_{CB}\pi_{BA}.$$

Now we will show that in order to compute an inverse limit we only need to use a cofinal subset of the directed (index) set.

Theorem 2.6. *Let $\{G_\alpha, \pi_{\alpha\beta}, A\}$ be an inverse system of groups, and let B be a cofinal subset of A. Then there is a homomorphism*

$$\pi_{AB} : \varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\} \to \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\}$$

Furthermore, π_{AB} is an isomorphism whose inverse is the projection map π_{BA}.

Proof. Take any $\alpha \in A$. Since B is cofinal, there is a $\beta \in B$ such that $\alpha < \beta$. Thus, consider the composition

$$f_\alpha := \pi_{\alpha\beta}\pi'_\beta : \varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\} \xrightarrow{\pi'_\beta} G_\beta \xrightarrow{\pi_{\alpha\beta}} G_\alpha$$

We write it as f_α since the election on β does not change the resulting map. In order to see this, consider $\beta_1, \beta_2 \in B$ such that $\alpha < \beta_i$, $i = 1, 2$, using that B is an ordered set, there exists a $\beta \in B$ such that $\beta_i < \beta$, and so the following diagram commutes

$$\begin{tikzcd}
 & & G_{\beta_1} \arrow[dr, "\pi_{\alpha\beta_1}"] & \\
\varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\} \arrow[urr, "\pi'_{\beta_1}"] \arrow[r, "\pi'_\beta"] \arrow[drr, "\pi'_{\beta_2}"'] & G_\beta \arrow[ur, "\pi_{\beta_1\beta}"] \arrow[rr, "\pi_{\alpha\beta}"] \arrow[dr, "\pi_{\beta_2\beta}"'] & & G_\alpha \\
 & & G_{\beta_2} \arrow[ur, "\pi_{\alpha\beta_2}"'] &
\end{tikzcd}$$

Note when $\alpha \in B$, we can simply take $\beta = \alpha$. Now let $\alpha_1, \alpha_2 \in A$, such that $\alpha_1 < \alpha_2$. Then there are $\beta_1, \beta_2 \in B$ such that $\alpha_i < \beta_i,\ i = 1, 2$. Using that B is a directed set, there exists $\beta \in B$ such that $\beta_i < \beta,\ i = 1, 2$. It follows that

$$\begin{aligned}
\pi_{\alpha_1\alpha_2} f_{\alpha_2} &= \pi_{\alpha_1\alpha_2}(\pi_{\alpha_2\beta_2}\pi'_{\beta_2}) \\
&= \pi_{\alpha_1\alpha_2}\pi_{\alpha_2\beta_2}\pi'_{\beta_2} \\
&= \pi_{\alpha_1\alpha_2}\pi_{\alpha_2\beta_2}(\pi_{\beta_2\beta}\pi'_{\beta}) \\
&= \pi_{\alpha_1\alpha_2}(\pi_{\alpha_2\beta_2}\pi_{\beta_2\beta})\pi'_{\beta} \\
&= \pi_{\alpha_1\alpha_2}\pi_{\alpha_2\beta}\pi'_{\beta} \\
&= (\pi_{\alpha_1\alpha_2}\pi_{\alpha_2\beta})\pi'_{\beta} \\
&= \pi_{\alpha_1\beta}\pi'_{\beta} \\
&= (\pi_{\alpha_1\beta_1}\pi_{\beta_1\beta})\pi'_{\beta} \\
&= \pi_{\alpha_1\beta_1}(\pi_{\beta_1\beta}\pi'_{\beta}) \\
&= \pi_{\alpha_1\beta_1}\pi'_{\beta_1} \\
&= f_{\alpha_1}
\end{aligned}$$

i.e, the following diagram commutes

$$\begin{tikzcd}
 & G_{\beta_2} \arrow[r, "\pi_{\alpha_2\beta_2}"] & G_{\alpha_2} \arrow[dd, "\pi_{\alpha_1\alpha_2}"] \\
\varprojlim\limits_{B}\{G_\beta, \pi_{\beta\gamma}\} \arrow[ur, "\pi'_{\beta_2}"] \arrow[r, "\pi'_\beta"] \arrow[dr, "\pi'_{\beta_1}"'] \arrow[urr, bend left, "f_{\alpha_2}"] \arrow[drr, bend right, "f_{\alpha_1}"'] & G_\beta \arrow[u, "\pi_{\beta_2\beta}"] \arrow[d, "\pi_{\beta_1\beta}"'] \arrow[ur, "\pi_{\alpha_2\beta}"] \arrow[dr, "\pi_{\alpha_1\beta}"'] & \\
 & G_{\beta_1} \arrow[r, "\pi_{\alpha_1\beta_1}"] & G_{\alpha_1}
\end{tikzcd}$$

Thus, $\{f_\alpha : \varprojlim\limits_{B}\{G_\beta, \pi_{\beta\gamma}\} \to G_\alpha\}_{\alpha \in A}$ is an inverse system of homomorphisms. Using the universal property of the inverse limit, for this inverse system of homomorphisms, there is a unique homomorphism

$$\pi_{AB} : \varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\} \to \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\}$$

such that $\pi_\alpha \pi_{AB} = f_\alpha$, i.e., the following diagram commutes

$$\begin{tikzcd}
& \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\} \arrow[d, "\pi_\alpha"] \\
\varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\} \arrow[ur, "\pi_{AB}"] \arrow[r, "f_\alpha"'] & G_\alpha
\end{tikzcd}$$

Now consider $\alpha \in A$ and $\beta \in B$ as above. Using the properties of π_{AB} and π_{BA}, we have that

$$\begin{aligned}
\pi_\alpha(\pi_{AB}\pi_{BA}) &= (\pi_\alpha \pi_{AB})\pi_{BA} \\
&= f_\alpha \pi_{BA} \\
&= (\pi_{\alpha\beta}\pi'_\beta)\pi_{BA} \\
&= \pi_{\alpha\beta}(\pi'_\beta \pi_{BA}) \\
&= \pi_{\alpha\beta}\pi_\beta \\
&= \pi_\alpha
\end{aligned}$$

and that

$$\begin{aligned}
\pi'_\beta(\pi_{BA}\pi_{AB}) &= (\pi'_\beta \pi_{BA})\pi_{AB} \\
&= \pi_\beta \pi_{AB} \\
&= f_\beta \\
&= \pi_{\beta\beta}\pi'_\beta \\
&= \pi'_\beta
\end{aligned}$$

This means that the following diagrams commute

$$\begin{tikzcd}
& & \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\} \arrow[dd, "\pi_\alpha"] \\
& \varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\} \arrow[ur, "\pi_{AB}"] \arrow[dr, "f_\alpha"] & \\
\varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\} \arrow[ur, "\pi_{BA}"] \arrow[rr, "\pi_\alpha"] \arrow[uurr, bend left, "\mathrm{Id}_{\varprojlim_{A}\{G_\alpha\}}"] & & G_\alpha
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& & \varprojlim_{B}\{G_\beta, \pi_{\beta\gamma}\} \arrow[dd, "f_\alpha"] \\
& \varprojlim_{A}\{G_\alpha, \pi_{\alpha\beta}\} \arrow[ur, "\pi_{BA}"] \arrow[dr, "\pi_\alpha"] & \\
\varprojlim_{B}\{G_\alpha, \pi_{\alpha\beta}\} \arrow[ur, "\pi_{AB}"] \arrow[rr, "f_\alpha"] \arrow[uurr, bend left, "\mathrm{Id}_{\varprojlim_{B}\{G_\beta\}}"] & & G_\alpha
\end{tikzcd}$$

Thus, using the uniqueness of the universal property of the inverse limit, we have that

$$\pi_{BA}\pi_{AB} = \mathrm{Id}_{\varprojlim_{B}\{G_\beta\}} \quad \text{and} \quad \pi_{AB}\pi_{BA} = \mathrm{Id}_{\varprojlim_{A}\{G_\alpha\}}$$

$\square$

Now let $\{G_\alpha, \pi_{\alpha\beta}, A\}$ and $\{H_\gamma, \kappa_{\gamma\eta}, B\}$ be two inverse systems of groups. If $\phi : B \to A$ is an order preserving map, that is, for every $\gamma, \eta \in B$ such that $\gamma < \eta$ then $\phi(\gamma) < \phi(\eta)$. For convenience of notation, we will write $\phi(\gamma) = \gamma',\ \phi(\eta) = \eta'$. Consider a family of homomorphisms $\{f_\gamma : G_{\gamma'} \to H_\gamma\}_{\gamma \in B}$ such that the follonwing diagram commutes

$$\begin{tikzcd} G_{\eta'} \arrow[r, "f_\eta"] \arrow[d, "\pi_{\gamma'\eta'}"'] & H_\eta \arrow[d, "\kappa_{\gamma\eta}"] \\ G_{\gamma'} \arrow[r, "f_\gamma"'] & H_\gamma \end{tikzcd}$$

whenever $\gamma < \eta$.

Definition 2.16. Such a family of homomorphisms $\{f_\gamma : G_{\gamma'} \to H_\gamma\}_{\gamma \in B}$ is called an *inverse system of homomorphisms of the system* $\{G_\alpha, \pi_{\alpha\beta}, A\}$ *into* $\{H_\gamma, \kappa_{\gamma\eta}, B\}$ *corresponding to the order preserving map* $\phi : B \to A$. We will denote this family by $\{f_\gamma : G_{\gamma'} \to H_\gamma\}$, when B is clear from context.

We will extend the result in Theorem 2.5 to an inverse system of homomorphisms.

Theorem 2.7. *Given* $\{f_\gamma : G_{\gamma'} \to H_\gamma\}$ *an inverse system of homomorphisms of the system* $\{G_\alpha, \pi_{\alpha\beta}, A\}$ *into* $\{H_\gamma, \kappa_{\gamma\eta}, B\}$ *corresponding an order preserving the map* $\phi : B \to A$. *There exists a unique homomorphism*

$$f : \varprojlim_A \{G_\alpha, \pi_{\alpha\beta}\} \to \varprojlim_B \{H_\gamma, \kappa_{\gamma\eta}\}$$

such that

$$\kappa_\gamma f = f_\gamma \pi_{\gamma'},$$

i.e., the following diagram commutes

$$\begin{tikzcd} \varprojlim_A \{G_\alpha, \pi_{\alpha\beta}\} \arrow[r, "f"] \arrow[d, "\pi_{\gamma'}"'] & \varprojlim_B \{H_\gamma, \kappa_{\gamma\eta}\} \arrow[d, "\kappa_\gamma"] \\ G_{\gamma'} \arrow[r, "f_\gamma"'] & H_\gamma \end{tikzcd}$$

Proof. Given $\gamma \in B$, consider the composition

$$\varprojlim_A \{G_\alpha, \pi_{\alpha\beta}\} \xrightarrow{\pi_{\gamma'}} G_{\gamma'} \xrightarrow{f_\gamma} H_\gamma$$

Using the universal property of the inverse limits, there is an unique f as desire. Furthermore, consider a subset $B' \subset B$ and define $A' := \phi(B') \subset A$. Then the following diagram commutes

$$\begin{tikzcd} \varprojlim_A \{G_\alpha, \pi_{\alpha\beta}\} \arrow[r, "f"] \arrow[d, "\pi_{A'A}"'] & \varprojlim_B \{H_\gamma, \kappa_{\gamma\eta}\} \arrow[d, "\kappa_{B'B}"] \\ \varprojlim_{A'} \{G_{\alpha'}, \pi_{\alpha'\beta'}\} \arrow[r, "f'"'] & \varprojlim_{B'} \{H_{\gamma'}, \kappa_{\gamma'\eta'}\} \end{tikzcd}$$

where f' is induced by the inverse system of homomorphisms $\{f_{\gamma'} : G_{\psi(\gamma')} \to H_{\gamma'}\}_{\gamma' \in B'}$ corresponding to $\psi : B' \to A'$. $\square$

Definition 2.17. The homomorphism f constructed above is called the *inverse limit of the inverse system of homomorphisms* $\{f_\gamma : G_{\gamma'} \to H_\gamma\}$.

Theorem 2.8. *Consider three inverse systems* $\{G_\alpha, \pi_{\alpha\beta}, A\}, \{H_\gamma, \kappa_{\gamma\eta}, B\}, \{K_\sigma, \mu_{\sigma\theta}, C\}$. *Let* $\psi : C \to B$ *and* $\phi : B \to A$ *be two order preserving maps. For convenience of notation, write* $\psi(\sigma) = \sigma', \phi(\gamma) = \gamma'$. *If* $\{f_\gamma : G_{\gamma'} \to H_\gamma\}_{\gamma \in B}$ *and* $\{g_\sigma : H_{\sigma'} \to K_\sigma\}_{\sigma \in C}$ *are inverse systems of homomorphisms corresponding to* ϕ *and* ψ. *Then*

$$\{g_\sigma f_{\sigma'} : G_{\sigma''} \to K_\sigma\}$$

is an inverse system of homomorphisms corresponding to $\phi\psi : C \to A$. *Furthermore, if* f, g, h *are the inverse limits of* $\{f_\gamma\}, \{g_\sigma\}, \{h_\sigma := g_\sigma f_{\sigma'}\}$, *respectively; then*

$$h = gf$$

Proof. Consider $\sigma < \theta$. Then we have the following diagram

$$\begin{tikzcd} G_{\theta''} \arrow[r, "f_{\theta'}"] \arrow[d, "\pi_{\sigma''\theta''}"'] & H_{\theta'} \arrow[r, "g_\theta"] \arrow[d, "\kappa_{\sigma'\theta'}"] & K_\theta \arrow[d, "\mu_{\sigma\eta}"] \\ G_{\sigma''} \arrow[r, "f_{\sigma'}"'] & H_{\sigma'} \arrow[r, "g_\sigma"'] & K_\sigma \end{tikzcd}$$

By hypothesis, the two squares are commutative, so the diagram is commutative. Thus $\{f_{\sigma'} g_\sigma : G_{\sigma''} \to K_\sigma\}$ is an inverse system of homomorphisms corresponding to $\phi\psi : C \to A$.

Using the following commutative diagram

$$\begin{tikzcd} \varprojlim_A \{G_\alpha, \pi_{\alpha\beta}\} \arrow[r, "f"] \arrow[d, "\pi_{\sigma''}"'] & \varprojlim_B \{H_\gamma, \kappa_{\gamma\eta}\} \arrow[r, "g"] \arrow[d, "\kappa_{\sigma'}"] & \varprojlim_C \{K_\sigma, \mu_{\sigma\theta}\} \arrow[d, "\mu_\sigma"] \\ G_{\sigma''} \arrow[r, "f_{\sigma'}"'] & H_{\sigma'} \arrow[r, "g_\sigma"'] & K_\sigma \end{tikzcd}$$

we have that

$$
\begin{aligned}
\mu_\sigma(gf) &= (\mu_\sigma g)f \\
&= (g_\sigma \kappa_{\sigma'})f \\
&= g_\sigma(\kappa_{\sigma'} f) \\
&= g_\sigma(f_{\sigma'}\pi_{\sigma''}) \\
&= (g_\sigma f_{\sigma'})\pi_{\sigma''} \\
&= h_\sigma \pi_{\phi\psi(\sigma)} \\
&= \mu_\sigma h
\end{aligned}
$$

Thus, using the uniqueness of the universal property of the inverse limit, we conclude that $h = gf$. □

2.4 Čech Homology definition

We will have a inverse systems of groups indexed by $\Gamma(X)$, with homomorphisms defined by the refinements. Using this we will be able to define the Čech homology of a closure space.

First, we need to show that in fact $\Gamma(X)$ is a directed set.

Lemma 2.9. *For a given closure space pair (X, A), the set of all interior covers $\Gamma(X, A)$ is a directed set.*

Proof. Let $(\mathscr{U}, \mathscr{U}_A), (\mathscr{V}, \mathscr{V}_A) \in \Gamma(X, A)$. First, define

$$\mathscr{W} := \{U \cap V |\, U \in \mathscr{U}, V \in \mathscr{V}\}$$

This is an interior cover of X, since

$$
\begin{aligned}
\bigcup_{W \in \mathscr{W}} i(W) &= \bigcup_{U \in \mathscr{U}} \bigcup_{V \in \mathscr{V}} i(U \cap V) \\
&= \bigcup_{U \in \mathscr{U}} \bigcup_{V \in \mathscr{V}} [i(U) \cap i(V)] \\
&= \bigcup_{U \in \mathscr{U}} \left[i(U) \cap \left[\bigcup_{V \in \mathscr{V}} i(V) \right] \right] \\
&= \bigcup_{U \in \mathscr{U}} [i(U) \cap X] \\
&= \bigcup_{U \in \mathscr{U}} i(U) \\
&= X
\end{aligned}
$$

Now, define $\mathscr{W}_A := \{U \cap V \,|\, U \in \mathscr{U}_A, V \in \mathscr{V}_A\}$. Using that

$$\begin{aligned}
\bigcup_{W \in \mathscr{W}_A} i(W) &= \bigcup_{U \in \mathscr{U}_A} \bigcup_{V \in \mathscr{V}_A} i(U \cap V) \\
&= \bigcup_{U \in \mathscr{U}_A} \bigcup_{V \in \mathscr{V}_A} [i(U) \cap i(V)] \\
&= \bigcup_{U \in \mathscr{U}_A} \left[i(U) \cap \left[\bigcup_{V \in \mathscr{V}_A} i(V) \right] \right] \\
&\supset \bigcup_{U \in \mathscr{U}} [i(U) \cap A] \\
&= \left[\bigcup_{U \in \mathscr{U}} i(U) \right] \cap A \\
&\supset A,
\end{aligned}$$

we conclude that the pair $(\mathscr{W}, \mathscr{W}_A)$ is an interior cover of the pair (X, A). Also note that $\mathscr{W}$ is a common refinement of both $\mathscr{U}$ and $\mathscr{V}$, i.e., $\mathscr{U} < \mathscr{W}$ and $\mathscr{V} < \mathscr{W}$, because for each $W \in \mathscr{W}$ there are $U \in \mathscr{U}$, $V \in \mathscr{V}$ such that $W = U \cap V$, and so $W \subset U$ and $W \subset V$. Similarly, we have that $\mathscr{W}_A$ is a common refinement of borh $\mathscr{U}_A$ and $\mathscr{V}_A$. Thus, we conclude that $(\mathscr{W}, \mathscr{W}_A) < (\mathscr{U}, \mathscr{U}_A)$ and $(\mathscr{W}, \mathscr{W}_A) < (\mathscr{V}, \mathscr{V}_A)$. □

Recall that for each $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$ there is a group $H_n(X, A; \mathscr{U}, \mathscr{U}_A)$, and for a refinement $(\mathscr{U}, \mathscr{U}_A) < (\mathscr{V}, \mathscr{V}_A)$ there is a homomorphism $\pi_{\mathscr{U}\mathscr{V}*} : H_n(X, A; \mathscr{V}, \mathscr{V}_A) \to H_n(X, A; \mathscr{U}, \mathscr{U}_A)$. Thus, $\{H_n(X, A; \mathscr{U}, \mathscr{U}_A), \pi_{\mathscr{U}\mathscr{V}*}, \Gamma(X, A)\}$ is an inverse system of groups.

Definition 2.18. The n^{th} *Čech homology group* is the inverse limit of the inverse system defined above, i.e.,

$$\check{H}_n(X, A) := \varprojlim_{\Gamma(X,A)} \{H_n(X, A; \mathscr{U}, \mathscr{U}_A), \pi_{\mathscr{U}\mathscr{V}*}\}$$

If $A = \emptyset$, then $\check{H}_n(X, A)$ is written as $\check{H}_n(X)$.

Observation 5. Even though $\Gamma(X)$ and $\Gamma(X, \emptyset)$ are different directed systems, we have that each $\mathscr{U} \in \Gamma(X)$ has a corresponding $(\mathscr{U}, \emptyset) \in \Gamma(X, \emptyset)$. Also, note that for any $(\mathscr{U}, \mathscr{U}_\emptyset) \in \Gamma(X, \emptyset)$ the cover $(\mathscr{U}, \emptyset)$ is a refinement of $(\mathscr{U}, \mathscr{U}_\emptyset)$. Thus, we can consider $\Gamma(X)$ as a cofinal subset of $\Gamma(X, \emptyset)$, and so if a limit process is over $\Gamma(X, \emptyset)$, then we will consider the limit over $\Gamma(X)$.

2.5 Direct limits

The notion of direct limit is dual to the inverse limit, in the sense that, categorically, a direct limit is an inverse limit in the opposite category, and vice-versa. An important difference, however, is that direct limits preserve exact sequences, which will allow us to define a Mayer Vietoris for cohomology and inverse limits do not.

Now, we will follow the structure of [12], but with a different take to the proofs, since they rely on an inductive argument and we will take a more categorical one.

Definition 2.19. A *direct system of groups* is a set of groups G_α, indexed by a directed set A, such that, for all $\alpha, \beta \in A$ with $\alpha > \beta$ there exists a homomorphism $\pi^{\alpha\beta} : G_\beta \to G_\alpha$. These homomorphisms satisfy the conditions

1. $\pi^{\alpha\alpha} = \mathrm{Id}_{G_\alpha}$ for all $\alpha \in A$.

2. $\pi^{\alpha\beta}\pi^{\beta\gamma} = \pi^{\alpha\gamma}$, whenever $\alpha > \beta > \gamma$.

We will denote this by $\{G_\alpha, \pi^{\alpha\beta}, A\}$. When context allows it, we will simply write $\{G_\alpha, \pi^{\alpha\beta}\}$, or $\{G_\alpha\}$.

Now remember that in the context of (abelian) groups, the direct sum of a collection of groups $\{G_\alpha\}_{\alpha\in A}$ is

$$\bigoplus_{\alpha\in A} G_\alpha := \left\{(g_\alpha) \in \prod_{\alpha\in A} G_\alpha \middle| g_\alpha = 0, \text{ but for finite many } \alpha \in A\right\} \subset \prod_{\alpha\in A} G_\alpha$$

along with the natural inclusions $\iota^\beta : G_\beta \hookrightarrow \bigoplus_{\alpha\in A} G_\alpha$ defined by $\iota^\beta(g_\beta) = (g_\alpha)_{\alpha\in A}$, where $g_\alpha = 0$ if $\alpha \neq \beta$.

Similar to the universal property of the product of groups, we have a corresponding property for the direct sum.

Theorem 2.10 (Universal Property of the coproduct)**.** *Given a group H and a collection of groups $\{G_\alpha\}_{\alpha\in A}$, indexed by a set A. If for each $\alpha \in A$ there is an homomorphism $f_\alpha : G_\alpha \to H$, then there exists a unique homomorphism $f : \bigoplus_{\alpha\in A} G_\alpha \to H$ such that*

$$f\iota^\alpha = f_\alpha,$$

i.e., the following diagram commutes

$$\begin{tikzcd}
\bigoplus_{\alpha\in A} G_\alpha \arrow[dr, "f"] & \\
G_\alpha \arrow[u, "\iota^\alpha"] \arrow[r, "f_\alpha"'] & H
\end{tikzcd}$$

Now, consider a direct system $\{G_\alpha, \pi^{\alpha\beta}, A\}$. Let R be the subgroup of $\bigoplus_{\alpha\in A} G_\alpha$ generated by elements of the form $x_\beta - \pi^{\alpha\beta}(x_\beta)$, for all $\alpha > \beta$. Define

$$G := \bigoplus_{\alpha\in A} G_\alpha / R$$

Definition 2.20. The *direct limit of the direct system* $\{G_\alpha, \pi^{\alpha\beta}, A\}$ is the group G defined as above. We'll denote it by $\underset{\overrightarrow{A}}{\lim}\{G_\alpha, \pi^{\alpha\beta}\}$, $\varinjlim\{G_\alpha, \pi^{\alpha\beta}\}$, or simply $\varinjlim\{G_\alpha\}$, when A or $\pi^{\alpha\beta}$ are clear from context.

Note that the limit process identifies the element $x_\beta \in G_\beta$ with the elements $\pi^{\alpha\beta}(x_\beta) \in G_\alpha$, whenever $\alpha > \beta$.

Definition 2.21. For each $\beta \in A$, there is a natural homomorphism $\pi^\beta : G_\beta \to \varinjlim\{G_\alpha, \pi^{\alpha\beta}\}$ corresponding to the composite

$$G_\beta \xrightarrow{\iota^\beta} \bigoplus_{\alpha\in A} G_\alpha \xrightarrow{p} \varinjlim\{G_\alpha, \pi^{\alpha\beta}\},$$

where ι^β is the natural inclusion and p is the natural projection. We say π^β is *the inclusion of G_β into* $\varprojlim\{G_\alpha\}$.

Theorem 2.11 (Universal Property of direct limits)**.** *Let $\{G_\alpha, \pi^{\alpha\beta}, A\}$ de a direct system of groups. Given a group H and homomorphisms $\{f_\alpha : G_\alpha \to H\}_{\alpha\in A}$ such that*

$$f_\beta = f_\alpha \pi^{\alpha\beta}$$

whenever $\alpha > \beta$. Then there exists a unique homomorphism $f : \varinjlim\{G_\alpha\} \to H$, such that

$$f\pi^\alpha = f_\alpha, \tag{2.4}$$

i.e., the following diagram commutes

$$\begin{tikzcd}
\varinjlim\{G_\alpha\} \arrow[dr, "f"] & \\
G_\alpha \arrow[u, "\pi^\alpha"] \arrow[r, "f_\alpha"'] & H
\end{tikzcd}$$

Proof. Using the universal property of the coproduct, there is a unique homomorphism

$$\hat{f} : \bigoplus_{\alpha\in A} G_\alpha \to H$$

such that $\hat{f} i^\alpha = f_\alpha$, i.e., $\hat{f}(x_\alpha) = f_\alpha(x_\alpha)$, for each $x_\alpha \in G_\alpha$. Consider $\beta \in A$ such that $\beta < \alpha$. Let

$x_\beta \in G_\beta$, then $\pi^{\alpha\beta}(x_\beta) \in G_\alpha$ and so

$$\begin{aligned}
\hat{f}(x_\beta - \pi^{\alpha\beta}(x_\beta)) &= \hat{f}(x_\beta) - \hat{f}(\pi^{\alpha\beta}(x_\beta)) \\
&= f_\beta(x_\beta) - f_\alpha(\pi^{\alpha\beta}(x_\beta)) \\
&= f_\beta(x_\beta) - (f_\alpha\pi^{\alpha\beta})(x_\beta)) \\
&= f_\beta(x_\beta) - f_\beta(x_\beta) \\
&= 0
\end{aligned}$$

since, by hypothesis, $f_\beta = f_\alpha\pi^{\alpha\beta}$. It follows that $R \subset \ker(\hat{f})$, with R as defined in 2.20. Using the universal property of the quotient group, we have a unique homomorphism $f : \underrightarrow{\lim}\{G_\alpha\} \to H$ such that for each $\beta \in A$ the following diagram commutes

$$\begin{tikzcd}
\underrightarrow{\lim}\{G_\alpha\} \arrow[dr, "f"] & \\
\bigoplus_{\alpha\in A} G_\alpha \arrow[u] \arrow[r, "\hat{f}"] & H \\
G_\beta \arrow[u] \arrow[ur, "f_\beta"'] \arrow[uu, bend left=60, "\pi^\beta"] &
\end{tikzcd}$$

$\square$

We will show an alternative construction for the direct limit, which will result more useful and eventually necessary in order to see that direct limits preserve exact sequences.

Consider the disjoint union of sets $\sqcup_{\alpha\in A} G_\alpha$. Each point $\sqcup_{\alpha\in A} G_\alpha$ can be thought as a pair (x_α, α) such that $x_\alpha \in G_\alpha$. Define a relation between these pairs by $(x_\alpha, \alpha) \sim (x_\beta, \beta)$ if there is $\delta > \alpha, \beta$ such that $\pi^{\delta\alpha}(x_\alpha) = \pi^{\delta\beta}(x_\beta)$. This is an equivalence relation:

- For each $\alpha \in A$, $\pi^{\alpha\alpha}(x_\alpha) = \pi^{\alpha\alpha}(x_\alpha)$, and so $(x_\alpha, \alpha) \sim (x_\alpha, \alpha)$. Thus, the relation is reflexive.
- Let $(x_\alpha, \alpha) \sim (x_\beta, \beta)$. This means that there is $\delta > \alpha, \beta$ such that $\pi^{\delta\alpha}(x_\alpha) = \pi^{\delta\beta}(x_\beta)$. Thus, $(x_\beta, \beta) \sim (x_\alpha, \alpha)$, i.e., the relation is symmetric.
- Let $(x_\alpha, \alpha) \sim (x_\beta, \beta)$ and $(x_\beta, \beta) \sim (x_\gamma, \gamma)$. By definition, there are $\delta > \alpha, \beta$ and $\lambda > \beta, \gamma$ such that

$$\pi^{\delta\alpha}(x_\alpha) = \pi^{\delta\beta}(x_\beta), \quad \text{and} \quad \pi^{\lambda\beta}(x_\beta) = \pi^{\lambda\gamma}(x_\gamma).$$

Since A is a directed set, there exists $\eta > \delta, \lambda$, and it follows that

$$\begin{aligned}\pi^{\eta\alpha}(x_\alpha) &= \pi^{\eta\delta}(\pi^{\delta\alpha}(x_\alpha)) \\ &= \pi^{\eta\delta}(\pi^{\delta\beta}(x_\beta)) \\ &= \pi^{\eta\beta}(x_\beta) \\ &= \pi^{\eta\lambda}(\pi^{\lambda\beta}(x_\beta)) \\ &= \pi^{\eta\lambda}(\pi^{\lambda\gamma}(x_\gamma)) \\ &= \pi^{\eta\gamma}(x_\gamma)\end{aligned}$$

Thus, the relation is transitive.

Let $\hat{G}$ be the set of the equivalence classes on $\bigsqcup_{\alpha\in A} G_\alpha$ with the equivalence relation described above, i.e.,

$$\hat{G} = \left(\bigsqcup_{\alpha\in A} G_\alpha\right)/\sim$$

Now we will describe a group operation on $\hat{G}$. Let $[x_\alpha, \alpha], [x_\beta, \beta] \in \hat{G}$, define

$$[x_\alpha, \alpha] + [x_\beta, \beta] := [\pi^{\delta\alpha}(x_\alpha) + \pi^{\delta\beta}(x_\beta), \delta]$$

for some $\delta \in A$ such that $\delta > \alpha, \beta$. In order to see that this operation is well defined, first we need to prove that the election of δ does not affect the result. Let $\delta_1, \delta_2 \in A$ such that $\delta_1 > \alpha, \beta$ and $\delta_2 > \alpha, \beta$. Using that A is a directed set, there is $\delta \in A$ such that $\delta > \delta_1, \delta_2$ and

$$\begin{aligned}\pi^{\delta\delta_1}(\pi^{\delta_1,\alpha}(x_\alpha) + \pi^{\delta_1\beta}(x_\beta)) &= \pi^{\delta\delta_1}\pi^{\delta_1\alpha}(x_\alpha) + \pi^{\delta\delta_1}\pi^{\delta_1\beta}(x_\beta) \\ &= \pi^{\delta\alpha}(x_\alpha) + \pi^{\delta\beta}(x_\beta) \\ &= \pi^{\delta\delta_2}\pi^{\delta_2\alpha}(x_\alpha) + \pi^{\delta\delta_2}\pi^{\delta_2\beta}(x_\beta) \\ &= \pi^{\delta\delta_2}(\pi^{\delta_2\alpha}(x_\alpha) + \pi^{\delta_2\beta}(x_\beta))\end{aligned}$$

Thus $\pi^{\delta_1\alpha}(x_\alpha) + \pi^{\delta_1\beta}(x_\beta) \sim \pi^{\delta_2\alpha}(x_\alpha) + \pi^{\delta_2\beta}(x_\beta)$.

Then we need to prove that the election of the representatives does not matter for the operation. Let $(x_{\alpha_1}, \alpha_1) \sim (x_{\alpha_2}, \alpha_2)$ and $(x_{\beta_1}, \beta) \sim (x_{\beta_2}, \beta)$. Then there are $\alpha > \alpha_1, \alpha_2$ and $\beta > \beta_1, \beta_2$ such that

$$\pi^{\alpha,\alpha_1}(x_{\alpha_1}) = \pi^{\alpha,\alpha_2}(x_{\alpha_2}), \quad \text{and} \quad \pi^{\beta,\beta_1}(x_{\beta_1}) = \pi^{\beta,\beta_2}(x_{\beta_2}).$$

Using that A is a directed set, there is $\delta > \alpha, \beta$. It follows that

$$\begin{aligned}\pi^{\delta\alpha_1}(x_{\alpha_1}) + \pi^{\delta\beta_1}(x_{\beta_1}) &= \pi^{\delta\alpha}(\pi^{\alpha\alpha_1}(x_{\alpha_1})) + \pi^{\delta\beta}(\pi^{\beta\beta_1}(x_{\beta_1})) \\ &= \pi^{\delta\alpha}(\pi^{\alpha\alpha_2}(x_{\alpha_2})) + \pi^{\delta\beta}(\pi^{\beta\beta_2}(x_{\beta_2})) \\ &= \pi^{\delta\alpha_2}(x_{\alpha_2}) + \pi^{\delta\beta_2}(x_{\beta_2})\end{aligned}$$

Thus, this operation is well defined. Furthermore, we have that:

- This operation is commutative, since

$$[x_\alpha, \alpha] + [x_\beta, \beta] = [\pi^{\delta\alpha}(x_\alpha) + \pi^{\delta\beta}(x_\beta), \delta] = [\pi^{\delta\beta}(x_\beta) + \pi^{\delta\alpha}(x_\alpha), \delta] = [x_\beta, \beta] + [x_\alpha, \alpha]$$

- For any $\alpha, \beta \in A$, $[0, \alpha] = [0, \beta]$, which is the identity element because

$$\pi^{\delta\alpha}(0) = \pi^{\delta\beta}(0) = 0$$

  and $(x_\alpha, \alpha) \sim (\pi^{\delta\alpha}(x_\alpha), \delta)$, for any $\delta > \alpha$.

- The inverse of $[x_\alpha, \alpha]$ is $[-x_\alpha, \alpha]$.

- For each $\alpha \in A$, there is a map $\tau^\alpha : G_\alpha \to \hat{G}$ defined by $\tau^\alpha(x_\alpha) = [x_\alpha, \alpha]$, which is the inclusion from G_α into $\hat{G}$. Furthermore, τ^α is an homomorphism, since

$$[x_\alpha, \alpha] + [y_\alpha, \alpha] = [\pi^{\alpha\alpha}(x_\alpha) + \pi^{\alpha\alpha}(y_\alpha), \alpha] = [x_\alpha + y_\alpha, \alpha]$$

- If $\alpha, \beta \in A$ are such that $\alpha > \beta$, then $\tau^\alpha\pi^{\alpha\beta} = \tau^\beta$, i.e., the following diagram commutes

$$\begin{tikzcd} & \hat{G} \\ G_\beta \arrow[ur, "\tau^\beta"] \arrow[r, "\pi^{\alpha\beta}"'] & G_\alpha \arrow[u, "\tau^\alpha"'] \end{tikzcd}$$

This construction is equivalent to the direct limit in the sense that they are isomorphic to each other. For this, we will show the following result.

Proposition 2.12. *Let $\{G_\alpha\}$ be a direct system of groups, and define $\hat{G}$ as above. If $\tau^\alpha : G_\alpha \to \hat{G}$ is the inclusion from G_α into $\hat{G}$, then $\hat{G}$ also satisfies the universal property of the direct limit.*

Proof. Given a group H and homomorphisms $\{f_\alpha : G_\alpha \to H\}_{\alpha\in A}$ such that

$$f_\beta = f_\alpha\pi^{\alpha\beta}$$

whenever $\alpha > \beta$. Define $\tilde{f} : \hat{G} \to H$ by $\tilde{f}([x_\alpha, \alpha]) = f_\alpha(x_\alpha)$. This is well defined since, if $(x_\alpha, \alpha) \sim (x_\beta, \beta)$, there is a $\delta > \alpha, \beta$ such that $\pi^{\delta\alpha}(x_\alpha) = \pi^{\delta\beta}(x_\beta)$, and so

$$f_\alpha(x_\alpha) = f_\delta(\pi^{\delta\alpha}(x_\alpha)) = f_\delta(\pi^{\delta\beta}(x_\beta)) = f_\delta(x_\delta)$$

Directly of the definition of τ^α and $\tilde{f}$, we have that $\tilde{f}\tau^\alpha = f_\alpha$. In order to see that $\tilde{f}$ is a homomorphism, note that, for any $\delta > \alpha, \beta$, we have

$$\begin{aligned}
\tilde{f}([\pi^{\delta\alpha}(x_\alpha) + \pi^{\delta\beta}(x_\beta), \delta]) &= f_\delta(\pi^{\delta\alpha}(x_\alpha) + \pi^{\delta\beta}(x_\beta)) \\
&= f_\delta(\pi^{\delta\alpha}(x_\alpha)) + f_\delta(\pi^{\delta\beta}(x_\beta)) \\
&= f_\alpha(x_\alpha) + f_\beta(x_\beta) \\
&= \tilde{f}([x_\alpha, \alpha]) + \tilde{f}([x_\beta, \beta])
\end{aligned}$$

Suppose there exists another $\tilde{f}' : \hat{G} \to H$ such that $\tilde{f}'\tau^\alpha = f_\alpha$. Then

$$\tilde{f}'([x_\alpha, \alpha]) = \tilde{f}'(\tau^{\alpha(x_\alpha)}) = f_\alpha(x_\alpha)$$

Thus, $\tilde{f}' = \tilde{f}$. $\square$

The universal property will give us the desired isomorphism.

Corollary. $\varinjlim\{G_\alpha\} \cong \hat{G}$

Proof. Using the universal property of direct limits, there are $\tau : \varinjlim\{G_\alpha\} \to \hat{G}$, and $\pi : \hat{G} \to \varinjlim\{G_\alpha\}$, such that $\tau\pi^\alpha = \tau^\alpha$, and $\pi\tau^\alpha = \pi^\alpha$, for each $\alpha \in A$. Note that $\tau\pi : \hat{G} \to \hat{G}$ satisfies

$$(\tau\pi)\tau^\alpha = \tau(\pi^\alpha) = \tau^\alpha, \tag{2.5}$$

i.e., the following diagram commutes

$$\begin{tikzcd}
\hat{G} \arrow[dr, dashed, "\pi"] \arrow[rrdd, bend left, "\mathrm{Id}_{\hat{G}}"] & & \\
& \varinjlim\{G_\alpha\} \arrow[dr, dashed, "\tau"] & \\
G_\alpha \arrow[uu, "\tau^\alpha"] \arrow[ur, dashed, "\pi^\alpha"] \arrow[rr, "\tau^\alpha"'] & & \hat{G}
\end{tikzcd}$$

Using the uniqueness of the universal property, we conclude that $\tau\pi = \mathrm{Id}_{\hat{G}}$. Similarly, we have that $\pi\tau = \mathrm{Id}_{\varinjlim\{G_\alpha\}}$. $\square$

The following lemmas will be used to prove that direct limits preserve exact sequences.

Lemma 2.13. *For each element $x \in \varinjlim\{G_\alpha\}$ there is a $\beta \in A$ and a $x_\beta \in G_\beta$ such that*

$$x = \pi^\beta(x_\beta)$$

Proof. Using the corollary 2.5, for each $x \in \varinjlim\{G_\alpha\}$, we have that $\tau(x) \in \hat{G}$. Therefore, there is $\beta \in A$ and $x_\beta \in G_\beta$ such that

$$\tau(x) = [x_\beta, \beta] = \tau^\beta(x_\beta)$$

Thus,

$$\pi^\beta(x_\beta) = \pi(\tau^\beta(x_\beta)) = \pi(\tau(x)) = x$$

□

Lemma 2.14. *For each $\beta \in A$ there is $\alpha > \beta$ such that $\ker(\pi^\beta) \subset \ker(\pi^{\alpha\beta})$,i.e., if $x_\beta \in G_\beta$ is such that $\pi^\beta(x_\beta) = 0$ there is $\alpha > \beta$ such that $\pi^{\alpha\beta}(x_\beta) = 0$.*

Proof. Consider π, τ as in the corollary 2.5. Let $x_\beta \in \ker(\pi^\beta)$. Then

$$[x_\beta, \beta] = \tau^\beta(x_\beta) = \tau(\pi^\beta(x_\beta)) = \tau(0) = [0, \beta']$$

where $\beta' \in A$ can be any index different from β. This means that $(x_\beta, \beta) \sim (0, \beta')$, and so there is $\alpha > \beta, \beta'$ such that

$$\pi^{\alpha\beta}(x_\beta) = \pi^{\alpha\beta'}(0) = 0$$

□

Now consider $B \subset A$, as directed sets. Take all the groups G_β, with $\beta \in B$. Note tha the corresponding restrictions maps are preserve, and so $\{G_\beta, \pi^{\beta\gamma}, B\}$ is a new direct system of groups. Remember that for each $\beta \in B$ there is a homomorphism $\pi^\beta : G_\beta \to \varinjlim_A\{G_\alpha, \pi^{\alpha\beta}\}$, such that $\pi^\gamma = \pi^\beta\pi^{\beta\gamma}$, whenever $\beta > \gamma$. Using the universal property of direct limits we have that there is a unique

$$\pi^{AB} : \varinjlim_B\{G_\beta, \pi^{\beta\gamma}\} \to \varinjlim_A\{G_\alpha, \pi^{\alpha\beta}\}$$

such that $\pi^{AB}\pi^{\beta'} = \pi^\beta$, where $\pi^{\beta'} : G_\beta \to \varinjlim_B\{G_\alpha\}$ is the natural inclusion of G_β into $\varinjlim_B\{G_\alpha\}$.

Definition 2.22. π^{AB} is called the *inclusion map of $\varinjlim_B\{G_\beta\}$ into $\varinjlim_A\{G_\alpha\}$*.

Remark. Let $C \subset B \subset A$ be directed sets. The uniqueness of the definition implies

$$\pi^{AC} = \pi^{AB}\pi^{BC}.$$

Theorem 2.15. *Let $\{G_\alpha, \pi^{\alpha\beta}, A\}$ be a direct system of groups, and let B be a cofinal set in A. Then there* *is a homomorphism*

$$\pi^{BA} : \varinjlim_{A}\{G_\alpha, \pi^{\alpha\beta}\} \to \varinjlim_{B}\{G_\beta, \pi^{\beta\gamma}\}$$

Furthermore, π^{BA} is an isomorphism whose inverse is π^{AB}.

Proof. Fix $\alpha \in A$. Since B is cofinal, there is $\beta \in B$ such that $\beta > \alpha$. Thus, consider the composition $f_\alpha := \pi^{\beta'}\pi^{\beta,\alpha}$ corresponding to the composite

$$G_\alpha \to G_{\beta_1} \to \varinjlim_{B}\{G_\beta, \pi^{\beta\gamma}\}$$

Note that the election of β does not affect the composition. In order to proof this , consider $\beta_1, \beta_2 \in B$ with $\alpha < \beta_i,\ i = 1, 2$. Using that B is a directed set, there is $\beta \in B$ such that $\beta > \beta_1, \beta_2$, and so

$$\pi^{\beta_1'}\pi^{\beta_1\alpha} = (\pi^{\beta'}\pi^{\beta\beta_1})\pi^{\beta_1\alpha} = \pi^{\beta'}\pi^{\beta\alpha} = \pi^{\beta'}(\pi^{\beta\beta_2}\pi^{\beta_2\alpha}) = \pi^{\beta_2'}\pi^{\beta_2\alpha},$$

i.e., the following diagram commutes

$$\begin{tikzcd}
 & G_{\beta_1} \arrow[dr, dashed, "\pi^{\beta\beta_1}"] \arrow[drr, "\pi^{\beta_1'}"] & & \\
G_\alpha \arrow[ur, "\pi^{\beta_1\alpha}"] \arrow[dr, "\pi^{\beta_2\alpha}"'] & & G_\beta \arrow[r, dashed, "\pi^{\beta}"] & \varinjlim_{B}\{G_{\beta'}, \pi^{\alpha\beta}\} \\
 & G_{\beta_2} \arrow[ur, dashed, "\pi^{\beta\beta_2}"] \arrow[urr, "\pi^{\beta_2'}"'] & &
\end{tikzcd}$$

Note that, when $\alpha \in B$, we can simply take $\beta = \alpha$. Also, this homomorphism satisfies that if $\alpha_1 > \alpha_2$, then there is $\beta \in B$ such that $\beta > \alpha_1, \alpha_2$. It follows that

$$\begin{aligned}
f_{\alpha_1}\pi^{\alpha_1\alpha_2} &= (\pi^{\beta'}\pi^{\beta\alpha_1})\pi^{\alpha_1\alpha_2} \\
&= \pi^{\beta'}\pi^{\beta\alpha_2} \\
&= f_{\alpha_2}
\end{aligned}$$

Using the universal property of the direct limit, there exists a unique homomorphism

$$\pi^{BA} : \varinjlim_{A}\{G_\alpha, \pi^{\alpha\beta}\} \to \varinjlim_{B}\{G_\beta, \pi^{\beta\gamma}\}$$

such that $\pi^{BA}\pi^\alpha = f_\alpha = \pi^{\beta'}\pi^{\beta\alpha}$, for each $\alpha \in A$.

Note that for each $\alpha \in A$, there is a $\beta \in B$ such that $\beta > \alpha$ and so

$$\begin{aligned}
(\pi^{AB}\pi^{BA})\pi^{\alpha} &= \pi^{AB} f_{\alpha} \\
&= \pi^{AB}(\pi^{\beta'}\pi^{\beta\alpha}) \\
&= \pi^{\beta}\pi^{\beta\alpha} \\
&= \pi^{\alpha},
\end{aligned}$$

i.e., the following diagram commutes

$$\begin{tikzcd}
\varprojlim_A\{G_\alpha, \pi_{\alpha\beta}\} \arrow[dr, "\pi^{BA}"] \arrow[ddrr, bend left, "\mathrm{Id}_{\varprojlim_A\{G_\alpha\}}"] & & \\
& \varprojlim_B\{G_\beta, \pi_{\beta\gamma}\} \arrow[dr, "\pi^{AB}"] & \\
G_\alpha \arrow[uu, "\pi^\alpha"] \arrow[ur, "f_\alpha"] \arrow[rr, "\pi^\alpha"] & & \varprojlim_A\{G_\alpha, \pi_{\alpha\beta}\}
\end{tikzcd}$$

Thus, using the uniqueness of the universal property of direct limits we have that $\pi^{AB}\pi^{BA} = \mathrm{Id}_{\varinjlim_A\{G_\alpha\}}$.

Now, for each $\beta \in B$ we can take $f_\beta = \pi^{\beta'}\pi^{\beta\beta} = \pi^{\beta'}$. It follows that

$$(\pi^{BA}\pi^{AB})\pi^{\beta'} = \pi^{BA}\pi^{\beta} = f_\beta = \pi^{\beta'},$$

and so, using the same argument as before, we have that $\pi^{BA}\pi^{AB} = \mathrm{Id}_{\varinjlim_B\{G_\beta\}}$. $\square$

Let $\{G_\alpha, \pi^{\alpha\beta}, A\}$ and $\{H_\gamma, \kappa^{\gamma\eta}, B\}$ be direct systems of groups. Consider $\phi : A \to B$ an order preserving map. For convenience of notation, we'll write $\phi(\alpha) = \alpha'$ and $\phi(\beta) = \beta'$. Let $\{f_\alpha : G_{\alpha'} \to H_\alpha,\ \alpha \in A\}$ be a family of homomorphisms such that the following diagram commutes

$$\begin{tikzcd}
G_\beta \arrow[r, "f_\beta"] \arrow[d, "\pi^{\alpha\beta}"'] & H_{\beta'} \arrow[d, "\kappa^{\alpha'\beta'}"] \\
G_\alpha \arrow[r, "f_\alpha"'] & H_{\alpha'}
\end{tikzcd}$$

whenever $\alpha > \beta$.

Definition 2.23. Such a family of homomorphisms $\{f_\alpha : G_\alpha \to H_{\alpha'},\ \alpha \in A\}$ is called an *direct system of homomorphisms of the system* $\{G_\alpha, \pi^{\alpha\beta}, A\}$ *into the system* $\{H_\gamma, \kappa^{\gamma\eta}, B\}$ *corresponding to the map* $\phi : A \to B$. We will denote this family by $\{f_\alpha : G_\alpha \to H_{\alpha'}\}$, when A is clear from context.

Theorem 2.16. *Let* $\{f_\alpha : G_\alpha \to H_{\alpha'}\}$ *be a direct system of homomorphisms of the system* $\{G_\alpha, \pi^{\alpha\beta}, A\}$ *into* $\{H_\gamma, \kappa^{\gamma\eta}, B\}$ *corresponding an order preserving the map* $\phi : A \to B$*. Then there exists a unique homomorphism*

$$f : \varinjlim_{A}\{G_\alpha, \pi^{\alpha\beta}\} \to \varinjlim_{B}\{H_\gamma, \kappa^{\gamma\eta}\}$$

such that, for each $\alpha \in A$*,*

$$f\pi^\alpha = \kappa^{\alpha'} f_\alpha$$

where $\alpha' = \phi(\alpha)$*, i.e., the following diagram commutes*

$$\begin{tikzcd}
G_\alpha \arrow[r, "f_\alpha"] \arrow[d, "\pi^\alpha"'] & H_{\alpha'} \arrow[d, "\kappa^{\alpha'}"] \\
\varinjlim_{A}\{G_\alpha, \pi^{\alpha\beta}\} \arrow[r, "f"'] & \varinjlim_{B}\{H_\gamma, \kappa^{\gamma\eta}\}
\end{tikzcd}$$

Proof. Given $\alpha \in A$, consider the composite $\kappa^{\alpha'} f_\alpha$ corresponding to

$$G_\alpha \xrightarrow{f_\alpha} H_{\alpha'} \xrightarrow{\kappa^{\alpha'}} \varinjlim_{B}\{H_\gamma, \kappa^{\gamma\eta}\}$$

Using the universal property of the inverse limits, there is a unique f as desire. □

Definition 2.24. This is called the *direct limit of the direct system of homomorphisms* $\{f_\alpha\}$.

Observation 6. Let $A' \subset A$. Define $B' := \phi(A')$. Then the following diagram commutes

$$\begin{tikzcd}
\varinjlim_{A'}\{G_{\alpha'}, \pi^{\alpha'\beta'}\} \arrow[r, "f'"] \arrow[d, "\pi^{AA'}"'] & \varinjlim_{B'}\{H_{\gamma'}, \kappa^{\gamma'\eta'}\} \arrow[d, "\kappa^{BB'}"] \\
\varinjlim_{A}\{G_\alpha, \pi^{\alpha\beta}\} \arrow[r, "f"'] & \varinjlim_{B}\{H_\gamma, \kappa^{\gamma\eta}\}
\end{tikzcd}$$

where f' is induced by the inverse system of homomorphisms $\{f_{\alpha'} : G_{\alpha'} \to H_{\alpha''}, A'\}$ corresponding to the restriction $\psi : A' \to B'$.

Theorem 2.17. *Let* $\{G_\alpha, \pi^{\alpha\beta}, A\}, \{H_\gamma, \kappa^{\gamma\eta}, B\}, \{L_\sigma, \mu^{\sigma\theta}, C\}$ *be three direct systems of groups. Let* $\phi : A \to B$ *and* $\psi : B \to C$ *be two order preserving maps. For convenience of notation, write* $\phi(\alpha) = \alpha', \psi(\gamma) = \gamma'$*. If* $\{f_\alpha : G_\alpha \to H_{\alpha'}\}$ *and* $\{g_\gamma : H_\gamma \to L_{\gamma'}\}$ *are systems of homomorphisms corresponding to* ϕ *and* ψ*, and if, for each* $\alpha \in A$*,* h_α *is the composite corresponding to* $G_\alpha \to H_{\alpha'} \to L_{\alpha''}$*. Then*

$$\{h_\alpha : G_\alpha \to L_{\alpha''}\}$$

is a direct system of homomorphisms corresponding to $\psi\phi : A \to C$*. Furthermore, if* f, g*, and* h *are the* *direct limits of* $\{f_\alpha\}, \{g_\gamma\}$*, and* $\{h_\alpha\}$*, respectively, then*

$$h = gf$$

Proof. Consider $\alpha > \beta$. Then we have the following diagram

$$\begin{tikzcd}
G_\beta \arrow[r, "f_\beta"] \arrow[d, "\pi^{\alpha\beta}"'] & H_{\beta'} \arrow[r, "g_{\beta'}"] \arrow[d, "\kappa^{\alpha'\beta'}"] & L_{\beta''} \arrow[d, "\mu^{\alpha''\beta''}"] \\
G_\alpha \arrow[r, "f_\alpha"'] & H_{\alpha'} \arrow[r, "g_{\alpha'}"'] & L_{\alpha''}
\end{tikzcd}$$

By hypothesis, the two squares are commutative, and so the diagram is commutative. Thus $\{h_\alpha := g_{\alpha'} f_\alpha : G_\alpha \to L_{\alpha''}\}$ is a direct system of homomorphisms corresponding to $\psi\phi : A \to C$.

Now, let $\alpha \in A$. Using the following commutative diagram

$$\begin{tikzcd}
G_\alpha \arrow[r, "f_\alpha"] \arrow[d, "\pi^\alpha"'] & H_{\alpha'} \arrow[r, "g_{\alpha'}"] \arrow[d, "\kappa^{\alpha'}"] & L_{\alpha''} \arrow[d, "\mu^{\alpha''}"] \\
\varinjlim_A \{G_\alpha, \pi^{\alpha\beta}\} \arrow[r, "f"'] & \varinjlim_B \{H_\gamma, \kappa^{\gamma\eta}\} \arrow[r, "g"'] & \varinjlim_C \{L_\sigma, \mu^{\sigma\theta}\}
\end{tikzcd}$$

we have that $(gf)\pi^\alpha = \mu^{\alpha''} h_\alpha = h\pi^\alpha$. Therefore, using the uniqueness of h, as the direct limit of homomorphisms, we have that in fact $gf = h$. $\square$

Theorem 2.18. *Consider the same conditions as above. Write* $A' := \phi(A) \subset B$ *and* $A'' = \psi(A') \subset C$*. Also, suppose that for each* $\alpha \in A$ *the sequence*

$$G_\alpha \xrightarrow{f_\alpha} H_{\alpha'} \xrightarrow{g'_\alpha} L_{\alpha''} \tag{2.6}$$

is exact, i.e., $\ker(g_{\alpha'}) = \mathrm{Im}\,(f_\alpha)$*. Then, the sequence*

$$\varinjlim_A \{G_\alpha\} \xrightarrow{f} \varinjlim_{A'} \{H'_\alpha\} \xrightarrow{g} \varinjlim_{A''} \{L_{\alpha''}\}$$

is exact, i.e., $\ker(g) = \mathrm{Im}\,(f)$*, where* f, g *are the direct limit of* $\{f_\alpha, A\}, \{g_{\alpha'}, A'\}$*.*

Proof. From Theorem 2.17, we have that the composite gf is the direct limit of the homomorphisms $\{0 = g_{\alpha'} f_\alpha, A\}$, and so $gf = 0$, i.e., $\mathrm{Im}\,(f) \subset \ker(g)$.

Now, we will prove the other inclusion. Let $y \in \ker(g) \subset \varinjlim_{A'} \{H'_\alpha, \pi^{\alpha\beta}\}$. Recall, from the Lemma 2.13, there are $\gamma' \in A'$ and $y_{\gamma'} \in H_{\gamma'}$ such that $y = \kappa^{\gamma'}(y_{\gamma'})$. It follows that

$$0 = g(y) = g(\kappa^{\gamma'}(y_{\gamma'})) = \mu^{\gamma''}(g_{\gamma'}(y_{\gamma'}))$$

Thus, $g_{\gamma'}(y_{\gamma'}) \in \ker(\mu^{\gamma''})$. Using the Lemma 2.14, there is a $\beta'' \in A''$ such that $\beta'' > \gamma''$ and $\ker(\mu^{\gamma''}) \subset \ker(\mu^{\beta''\gamma''})$. Thus, we have that

$$0 = \mu^{\beta''\gamma''}(g_{\gamma'}(y_{\gamma'})) = g_{\beta'}(\kappa^{\beta'\gamma'}(y_{\gamma'})),$$

and so $\kappa^{\beta'\gamma'}(y_{\gamma'}) \in \ker(g'_{\beta})$. Using the exactness at $H_{\beta'}$ in the sequence (2.6) for $\beta \in A$, there is $x_\beta \in G_\beta$ such that $f_\beta(x_\beta) = \kappa^{\beta',\gamma'}(y_{\gamma'})$. If we define $x = \pi^\beta(x_\beta) \in \varinjlim_{A}\{G_\alpha\}$, then we have that

$$f(x) = f(\pi^\beta(x_\beta)) = \kappa^{\beta'}(f_\beta(x_\beta)) = \kappa^{\beta'}(\kappa^{\beta'\gamma'}(y_{\gamma'})) = \kappa^{\gamma'}(y_{\gamma'}) = y$$

Thus, $\ker(g) \subset \mathrm{Im}\,(f)$. □

2.6 Čech Cohomology definition

We fixed a coefficient group for the simplicial cohomology, which for convenience it will be omitted. Let (X, A) be a pair. We have shown in the Lemma 2.9 that $\Gamma(X)$ is a directed set. Recall that for a given $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$, there is a simplicial pair $(K_{\mathscr{U}}, L_{\mathscr{U}_A})$, where $K_{\mathscr{U}}$ is the nerve of $\mathscr{U}$ and $L_{\mathscr{U}_A}$ is the subcomplex of $K_{\mathscr{U}}$ associated with the subspace A. Also, if $(\mathscr{V}, \mathscr{V}_A) \in \Gamma(X, A)$ is a refinement of $(\mathscr{U}, \mathscr{U}_A)$, i.e., $(\mathscr{U}, \mathscr{U}_A) < (\mathscr{V}, \mathscr{V}_A)$, then there is a simplicial map $\pi^1_{\mathscr{U}\mathscr{V}} : (K_{\mathscr{V}}, L_{\mathscr{V}_A}) \to (K_{\mathscr{U}}, L_{\mathscr{U}_A})$, and so

$$\pi^*_{\mathscr{V}\mathscr{U}} : H^n(K_{\mathscr{U}}, L_{\mathscr{U}_A}) \to H^n(K_{\mathscr{V}}, L_{\mathscr{V}_A})$$

is the induced homomorphism on the n^{th} cohomology groups. Write

$$H^n(X, A; \mathscr{U}) = H^n(K_{\mathscr{U}}, L_{\mathscr{U}_A})$$

Thus, $\{H^n(X, A; \mathscr{U}, \mathscr{U}_A), \pi^*_{\mathscr{V}\mathscr{U}}, \Gamma(X, A)\}$ is a direct system of groups.

The definition of the Çech cohomology is similar to the Čech homology, but instead of taking the inverse limit we take the direct limit of the directed system described above.

Definition 2.25. Write $\check{H}^n(X, A) = \varinjlim_{\Gamma(X,A)} \{H^n(X, A; (\mathscr{U}, \mathscr{U}_A)), \pi^*_{\mathscr{U}'\mathscr{U}}\}$. We call this the n^{th} *Čech cohomology group of* (X, A). If $A = \emptyset$, we write $\check{H}^n(X)$.

Chapter 3

Eilenberg-Steenrod Axioms

In 1945, Eilenberg and Steenrod [6] defined the axioms for homology as a way to give a more natural language for the homology groups in order to simplify their use. One should note that they lacked the definition of functor and natural transformation, but the notions appear as part of the axioms. As a part of their work, they sought to characterize different homology theories. In particular, two homology theories that satisfy the axioms and are isomorphic for the one point space, are isomorphic for any simplicial complex [6].

In this chapter, our main interest is to prove that the Čech (co)homology we defined in the previous chapter satisfies the functoriality, homotopy invariance and excision properties, but since the remaining properties are easy to establish, we will prove them as well. A first proof of the Eilenberg-Steenrod axioms for Čech homology on topological spaces was given by Dowker in 1952 [4]. The treatment given here is based on the books [12], [5], and [10].

Definition 3.1. *A closure space pair* $(X, A; \mathrm{c})$ is a set pair (X, A), where (X, c) is a closure space and $A \subset X$ is endowed with the subspace closure, which is defined by

$$c_A(U) := c(U) \cap A, \text{ for } U \subset A$$

We will refer to the closure space pair by (X, A), when c is understood. Also, if $A = \emptyset$, we will write the pair $(X, \emptyset)$ just as X.

Definition 3.2. Given two closure space pairs $(X, A; \mathrm{c}_X)$ and $(Y, B; \mathrm{c}_Y)$ and a function between set pairs $f : (X, A) \to (Y, B)$, i.e., $f(A) \subset B$. If $f : X \to Y$ is a continuous function, we say that $f : (X, A) \to (Y, B)$ is continuous.

Remark. If $f : (X, A) \to (Y, B)$ is continuous, then $f|_A \colon A \to B$ is a continuous function.

Proof. For any $C \subset A$, we have that

$$\begin{aligned} f\left(c_A\left(C\right)\right) &= f\left(c_X\left(C\right) \cap A\right) \\ &\subset f\left(c_X\left(C\right)\right) \cap f\left(A\right) \\ &\subset c_Y\left(f\left(C\right)\right) \cap B \\ &= c_B\left(f\left(C\right)\right) \end{aligned}$$

$\square$

We'll denote the category of closure space pairs by **Cl**. In the following chapter, we will consider the closure space $I = [0,1]$ with the usual topology. Also, given a closure space pair (X, A), we will write the closure space $(X, A) \times I = (X \times I, A \times I)$ with the product closure.

Since the constant functions $c_0, c_1 : (X, A) \to I$, defined by $c_0(x) = 0$ and $c_1(x) = 1$, and the identity $\mathrm{Id}_X : (X, A) \to (X, A)$ are continuous, then we have that $g_0, g_1 : (X, A) \to (X, A) \times I$ defined by

$$g_0(x) = (x, 0) \quad \text{and} \quad g_1(x) = (x, 1) \tag{3.1}$$

are continuous.

Observation 7. The category **Cl** is an example of an *admissible category for (co)homology theory* [5].

Definition 3.3. Let $f_0, f_1 : (X, A) \to (Y, B)$ be two continuous maps. We say that they are *homotopic in **Cl*** if there is a continuous function

$$H : (X, A) \times I \to (Y, B)$$

such that $f_0 = Hg_0$ and $f_1 = Hg_1$, with g_0, g_1 defined in (3.1), i.e.,

$$f_0(x) = H(x, 0) \quad \text{and} \quad f_1(x) = H(x, 1)$$

We will denote by $f_0 \sim f_1$ when the functions are homotopic, and we say H is a *homotopy*.

Let $\{H_n : \mathbf{Cl} \to \mathbf{Ab}\}$ be a sequence of functors from the category of closure space pairs **Cl** to the category of Abelian groups **Ab**, and let $\delta_n(X, A) : H_n(X, A) \to H_{n-1}(A)$ be a natural transformation, which we will call the boundary map. The Eilenberg-Steenrod axioms, as defined in [5] for admissible categories, are:

1. (Homotopy Invariance): If $f, g : (X, A) \to (Y, B)$ are homotopic maps in **Cl**, then the induced maps on (co)homology are the same.

2. (Exactness): Given a pair (X, A) with inclusions $\iota : A \to X$ and $j : X \to (X, A)$. For homology, there are homomorphism ∂_n such that the sequence

$$\ldots \to H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \ldots$$

is exact, and ∂ commutes with homomorphisms induced by continuous maps.

For cohomology, there are homomorphisms δ such that the sequence

$$\ldots \leftarrow H^n(A) \xleftarrow{\iota_*} H^n(X) \xleftarrow{j_*} H^n(X, A) \xleftarrow{\delta} H^{n-1}(A) \leftarrow \ldots$$

is exact, and δ commutes with homomorphisms induced by continuous maps.

3. (Dimension): If P is one-point space. For homology,

$$H_n(P) \cong \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0 \end{cases}$$

For cohomology,

$$H^n(P) \cong \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0 \end{cases}$$

4. (Excision): For a pair (X, A), if $U \subset X$ is such that $c\,(U) \subset i\,(A)$. Let $\iota : (X \setminus U, A \setminus U) \to (X, A)$ be the natural inclusion. Then the induced homomorphism in (co)homology isomorphisms.

First we will prove that the Čech homology and cohomology groups we defined on 2.18 and 2.25 are homology and cohomology theories. Thus, we need to prove these groups are functorial.

3.1 Functoriality

We will first show that there exists homomorphisms at the level of interior covers. These homomorphisms will define direct and indirect systems of homomorphisms for the case on cohomology and homology, respectively.

Lemma 3.1. *Let $f : (X, A) \to (Y, B)$ be a continuous map between closure space pairs. If $\Gamma(X, A)$ and $\Gamma(Y, B)$ are the sets of interior covers of (X, A) and (Y, B), respectively. Then there is an induced order preserving map $f^{-1} : \Gamma(Y, B) \to \Gamma(X, B)$ defined by $f^{-1}\,(\mathscr{U}, \mathscr{U}_A) := (f^{-1}\,(\mathscr{U}), f^{-1}\,(\mathscr{U}_A))$, where*

$$f^{-1}\,(\mathscr{U}) := \{f^{-1}\,(U) \mid U \in \mathscr{U}\} \quad \textit{and} \quad f^{-1}\,(\mathscr{U}_A) := \{f^{-1}\,(U) \mid U \in \mathscr{U}_A\}$$

Proof. First, fix an interior cover $(\mathscr{U}, \mathscr{U}_B) \in \Gamma(Y, B)$ of (Y, B). We will show that $f^{-1}(\mathscr{U}, \mathscr{U}_B)$ is an interior cover of (X, A).

Recall from 1.2, that for any $U \subset Y$, $f^{-1}(i_Y(U)) \subset i_X(f^{-1}(U))$. It follows that

$$\begin{aligned} X &= f^{-1}(Y) \\ &= f^{-1}\left(\bigcup_{U \in \mathscr{U}} i_Y(U)\right) \\ &= \bigcup_{U \in \mathscr{U}} f^{-1}(i_Y(U)) \\ &\subset \bigcup_{U \in \mathscr{U}} i_X\left(f^{-1}(U)\right) \end{aligned}$$

Therefore, we have that in fact $f^{-1}(\mathscr{U})$ is an interior cover of X. Now, since $f(A) \subset B$, we have that

$$\begin{aligned} A &\subset f^{-1}(B) \\ &\subset f^{-1}\left(\bigcup_{U \in \mathscr{U}_B} i_Y(U)\right) \\ &= \bigcup_{U \in \mathscr{U}_B} f^{-1}(i_Y(U)) \\ &\subset \bigcup_{U \in \mathscr{U}_B} i_X\left(f^{-1}(U)\right) \end{aligned}$$

Thus, $(f^{-1}(\mathscr{U}), f^{-1}(\mathscr{U}_A))$ is an interior cover of (X, A).

Now, we need to show that f^{-1} is an order preserving map. Let $(\mathscr{U}, \mathscr{U}_B), (\mathscr{V}, \mathscr{V}_B) \in \Gamma(Y, B)$ such that $(\mathscr{U}, \mathscr{U}_B) < (\mathscr{V}, \mathscr{V}_B)$, i.e., for each $V \in \mathscr{V}$ there is $U \in \mathscr{U}$ such that $V \subset U$, and for each $V \in \mathscr{V}_B$ there is $U \in \mathscr{U}_B$ such that $V \subset U$. Since $f^{-1}(V) \subset f^{-1}(U)$, we have that $f^{-1}(\mathscr{U}) < f^{-1}(\mathscr{V})$, and similarly $f^{-1}(\mathscr{U}_B) < f^{-1}(\mathscr{V}_B)$, i.e., $f^{-1}(\mathscr{U}, \mathscr{U}_B) < f^{-1}(\mathscr{V}, \mathscr{V}_B)$. Therefore, f^{-1} is in fact an order preserving map. $\square$

Proposition 3.2. *Let $f : (X, A) \to (Y, B)$ be a continuous map between closure space pairs, and $(\mathscr{U}, \mathscr{U}_B) \in \Gamma(Y, B)$, an interior cover of (Y, B). Consider $(\mathscr{U}', \mathscr{U}'_A) := f^{-1}(\mathscr{U}, \mathscr{U}_B)$, which is an interior cover of (X, A). If $(K_{\mathscr{U}}, L_{\mathscr{U}_B})$ is the simplicial pair corresponding to the nerve of $\mathscr{U}$ and the subcomplex of $K_{\mathscr{U}}$ corresponding to $B \subset Y$; and $(K_{\mathscr{U}'}, L_{\mathscr{U}'_A})$ is the simplicial pair corresponding to the nerve of $\mathscr{U}'$ and the subcomplex of $K_{\mathscr{U}'}$ corresponding to $A \subset X$. Then there exists a simplicial map*

$$f^1_{\mathscr{U}} : (K_{\mathscr{U}'}, L_{\mathscr{U}'_A}) \to (K_{\mathscr{U}}, L_{\mathscr{U}_B})$$

Proof. In order to construct the simplicial map, we will define the map in the vertices, then we extend it by linearity. Given U' a vertex in $K_{\mathscr{U}'}$, using the definition of $\mathscr{U}'$, there exists a $U \in \mathscr{U}$, which may not be unique, such that $U' = f^{-1}(U)$. If we fix a choice of U, then we can define $f^1_{\mathscr{U}}(U') := U$.

In order to verify that we can extend $f^1_{\mathscr{U}}$ to a simplicial map, let $U'_0, \ldots, U'_n$ be vertices of a simplex in $K_{\mathscr{U}'}$ and let $U_0, \ldots, U_n$ be their respective images under $f^1_{\mathscr{U}}$. By definition of the nerve of a cover, we have that $U'_0 \cap \ldots \cap U'_n \neq \emptyset$, and so

$$\emptyset \neq f(U'_0 \cap \ldots \cap U'_n) \subset f(U'_0) \cap \ldots \cap f(U'_n) \subset U_0 \cap \ldots \cap U_n,$$

since $f(U'_i) = f(f^{-1}(U_i)) \subset U_i$. It follows that U_i are vertices of a simplex in $K_{\mathscr{U}}$. Therefore $f^1_{\mathscr{U}}$ can be extended to a simplicial map.

Now we will show that $f^1_{\mathscr{U}}(L_{\mathscr{U}'_A}) \subset L_{\mathscr{U}_B}$. Let $U'_0, \ldots, U'_n$ are vertices of a simplex in $L_{\mathscr{U}'_A}$, i.e., they are vertices of a simplex in $K_{\mathscr{U}'}$ that satisfy $U'_0 \cap \ldots \cap U'_n \cap A \neq \emptyset$. Using that $f(A) \subset B$ and taking $U_0, \ldots, U_n$ as above, we have that

$$\emptyset \neq f(U'_0 \cap \ldots \cap U'_n \cap A) \subset f(U'_0) \cap \ldots \cap f(U'_n) \cap f(A) \subset U_0 \cap \ldots \cap U_n \cap B$$

It follows that the U_i are vertices of a simplex in $L_{\mathscr{U}_B}$. So the simplicial map $f^1_{\mathscr{U}}$ constructed before is a map from the pair $(K_{\mathscr{U}'}, L_{\mathscr{U}'_A})$ to the pair $(K_{\mathscr{U}}, L_{\mathscr{U}_B})$. ☐

Now, we will prove that the choice made in the construction of $f^1_{\mathscr{U}}$ doesn't affect the induced homomorphism on homology groups.

Lemma 3.3. *Let $f : (X, A) \to (Y, B)$ be a continuous function and let $(\mathscr{U}, \mathscr{U}_B)$ be a interior cover of (Y, B), and let $f^1_{\mathscr{U}}, f^2_{\mathscr{U}}$ be defined as above, but making different choices for each map $f^1_{\mathscr{U}}, f^2_{\mathscr{U}}$. Then $f^1_{\mathscr{U}}, f^2_{\mathscr{U}}$ are contiguous, as maps of simplicial pairs.*

Proof. Let $U'_0, \ldots, U'_n$ be vertices of a simplex in $K_{\mathscr{U}'}$, and let $U_0, \ldots, U_n$ and $V_0, \ldots, V_n$ be their respective images under $f^1_{\mathscr{U}}$ and $f^2_{\mathscr{U}}$, i.e., $f^{-1}(U_i) = f^{-1}(V_i) = U'_i$. It follows that

$$f^{-1}(U_0 \cap \ldots \cap U_n \cap V_0 \cap \ldots \cap V_n) = f^{-1}(U_0) \cap \ldots \cap f^{-1}(U_n) \cap f^{-1}(V_0) \cap \ldots \cap f^{-1}(V_n) = U'_0 \cap \ldots \cap U'_n \neq \emptyset$$

and so $U_0 \cap \ldots \cap U_n \cap V_0 \cap \ldots \cap V_n \neq \emptyset$. Using the definition of the nerve of a cover, we have that $U_0, \ldots, U_n, V_0, \ldots, V_n$ are vertices of a simplex of $K_{\mathscr{U}}$. Therefore, for any simplex $S \in K_{\mathscr{U}'}$, the corresponding images $f^1_{\mathscr{U}}(S)$ and $f^2_{\mathscr{U}}(S)$ are contained in some simplex of $K_{\mathscr{U}}$. Furthermore, if the U'_i are vertices of a simplex in $L_{\mathscr{U}'_A}$, then

$$\emptyset \neq U'_0 \cap \ldots \cap U'_n \cap A = f^{-1}(U_0 \cap \ldots \cap U_n \cap V_0 \cap \ldots \cap V_n \cap B)$$

By a similar reasoning, we have that $U_0, \ldots, U_n, V_0, \ldots, V_n$ are vertices of a simplex in $L_{\mathscr{U}_B}$. Thus, for any simplex S in $L_{\mathscr{U}'_A}$, $f^1_{\mathscr{U}}(S)$ and $f^2_{\mathscr{U}}(S)$ are contained in some simplex of $L_{\mathscr{U}_B}$. This proof that in fact $f^1_{\mathscr{U}}, f^2_{\mathscr{U}}$ are contiguous as maps of pairs. ☐

Since contiguous simplicial maps induced the same homomorphism on (co)homology groups, using lemma 2.2, we have that$f$ induces well-defined homomorphisms in homology

$$f_{\mathscr{U}*}: H_n(X, A; \mathscr{U}', \mathscr{U}'_A) \to H_n(Y, B; \mathscr{U}, \mathscr{U}_B)$$

and in cohomology

$$f^*_{\mathscr{U}}: H_n(Y, B; \mathscr{U}, \mathscr{U}_B) \to H_n(X, A; \mathscr{U}', \mathscr{U}'_A).$$

Definition 3.4. We call $f_{\mathscr{U}*}$ and $f^*_{\mathscr{U}}$ *the induced homomorphisms associated with the interior cover $\mathscr{U}$ and the continuous map f for homology and cohomology*, respectively.

Observation 8. If $\mathrm{Id}_X : (X, A) \to (X, A)$ is the identity function and $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$. Then $(\mathrm{Id}_X)_{\mathscr{U}*}$ and $(\mathrm{Id}_X)^*_{\mathscr{U}}$ are the identity.

Now we will prove the induced homomorphisms respect the composition of functions, given suitable interior covers.

Theorem 3.4. *Let $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ be continuous and let $(\mathscr{W}, \mathscr{W}_C) \in \Gamma(Z, C)$, an interior cover of (Z, C). If we define $(\mathscr{V}, \mathscr{V}_B) := g^{-1}(\mathscr{W}, \mathscr{W}_C)$, then we can define the induced simplicial maps such that*

$$(gf)^1_{\mathscr{W}} = g^1_{\mathscr{W}} f^1_{\mathscr{V}}.$$

Proof. For convenience write $h = gf$. From proposition 3.2, there exists induced simplicial maps $f^1_{\mathscr{V}}$ and $g^1_{\mathscr{W}}$. Write $(\mathscr{U}, \mathscr{U}_A) := f^{-1}(\mathscr{V}, \mathscr{V}_B)$. Given U a vertex of $K_{\mathscr{U}}$, we have that $V := f^1_{\mathscr{V}}(U)$ is a vertex of $K_{\mathscr{V}}$ such that $U = f^{-1}(V)$, and $W := g^1_{\mathscr{W}}(V)$ is a vertex of $K_{\mathscr{W}}$ such that $g^{-1}(W) = V$. It follows that

$$U = f^{-1}\left(g^{-1}(W)\right) = (gf)^{-1}(W) = h^{-1}(W),$$

and so we can define $h^1_{\mathscr{W}}(U) := W$. Therefore, we have that

$$h^1_{\mathscr{W}} = g^1_{\mathscr{W}} f^1_{\mathscr{V}}.$$

□

We have shown there are induced homomorphisms, which satisfy functorial properties, given an interior cover. Now we will show these homomorphisms define inverse and direct systems for homology and cohomology, respectively. Thus, we will prove the following lemma.

Lemma 3.5. *Let $f : (X, A) \to (Y, B)$ be continuous, and let $(\mathscr{U}, \mathscr{U}_B), (\mathscr{V}, \mathscr{V}_B) \in \Gamma(Y)$. Define $(\mathscr{U}', \mathscr{U}'_A) := f^{-1}(\mathscr{U}, \mathscr{U}_B), (\mathscr{V}', \mathscr{V}'_A) := f^{-1}(\mathscr{V}, \mathscr{V}_B)$. If $(\mathscr{U}, \mathscr{U}_B) < (\mathscr{V}, \mathscr{V}_B)$, then the following* *diagram of simplicial pairs commutes*

$$\begin{tikzcd}
(K_{\mathscr{V}'}, L_{\mathscr{V}'_A}) \arrow[r, "f^1_{\mathscr{V}}"] \arrow[d, "\pi^1_{\mathscr{U}'\mathscr{V}'}"'] & (K_{\mathscr{V}}, L_{\mathscr{V}_B}) \arrow[d, "\pi^1_{\mathscr{U}\mathscr{V}}"] \\
(K_{\mathscr{U}'}, L_{\mathscr{U}'_A}) \arrow[r, "f^1_{\mathscr{U}}"'] & (K_{\mathscr{U}}, L_{\mathscr{U}_B})
\end{tikzcd}$$

Proof. Using Lemma 3.1 and that $(\mathscr{U}, \mathscr{U}_B) < (\mathscr{V}, \mathscr{V}_B)$, we have that $(\mathscr{U}', \mathscr{U}'_A) < (\mathscr{V}', \mathscr{V}'_A)$. Now let $V' \in \mathscr{V}'$, then there exists $V \in \mathscr{V}$ such that $V' = f^{-1}(V)$. Using that $\mathscr{V}$ is a refinement of $\mathscr{U}$, there exists $U \in \mathscr{U}$ such that $V \subset U$. If we define $U' := f^{-1}(U)$, then we have that

$$V' = f^{-1}(V) \subset f^{-1}(U) = U'$$

Therefore, we can define the simplicial maps $f^1_{\mathscr{V}}(V') := V$, $\pi^1_{\mathscr{U}\mathscr{V}}(V) := U$, $f^1_{\mathscr{U}}(U') := U$, and $\pi^1_{\mathscr{U}'\mathscr{V}'}(V') := U'$. It follows that for all vertices of $K_{\mathscr{V}'}$, we can define the maps such that

$$\pi^1_{\mathscr{U}\mathscr{V}} f^1_{\mathscr{V}} = f^1_{\mathscr{U}} \pi^1_{\mathscr{U}'\mathscr{V}'} \tag{3.2}$$

After extending by linearity, we have the equation (3.2) holds for the whole complex $K_{\mathscr{V}'}$. ☐

In order to prove that the Čech homology and cohomology we defined is a functor, we will use the following theorem.

Theorem 3.6. *Let $f : (X, A) \to (Y, B)$ be a continuous function. Then there exists unique homomorphisms*

$$f_* : \check{H}_n(X, A) \to \check{H}_*(Y, B) \quad \textit{and} \quad f_* : \check{H}^n(Y, B) \to \check{H}^*(X, A)$$

such that for all $(\mathscr{U}, \mathscr{U}_B) \in \Gamma(Y)$ the following diagrams commute

$$\begin{tikzcd}
\check{H}_n(X, A) \arrow[r, "f_*"] \arrow[d, "\pi_{\mathscr{U}'*}"'] & \check{H}_n(Y, B) \arrow[d, "\pi_{\mathscr{U}*}"] \\
H_n(X, A; \mathscr{U}', \mathscr{U}'_A) \arrow[r, "f_{\mathscr{U}*}"'] & H_n(Y, B; \mathscr{U}, \mathscr{U}_B)
\end{tikzcd} \tag{3.3}$$

and

$$\begin{tikzcd}
H^n(Y, B; \mathscr{U}, \mathscr{U}_B) \arrow[r, "f^*_{\mathscr{U}}"] \arrow[d, "\pi^*_{\mathscr{U}}"'] & H^n(X, A; \mathscr{U}', \mathscr{U}'_A) \arrow[d, "\pi^*_{\mathscr{U}'}"] \\
\check{H}^n(Y, B) \arrow[r, "f_*"'] & \check{H}^n(X, A)
\end{tikzcd} \tag{3.4}$$

where $\pi_{\mathscr{U}*}, \pi_{\mathscr{U}'*}$ are the natural projections from the inverse limits, and $f^*_{\mathscr{U}}, \pi^*_{\mathscr{U}'}$ are the natural inclusions from the direct limits.

Proof. Recall the definitions of the Čech homology and cohomology as inverse and direct limits, respectively. We will use systems of homomorphisms in order to define the desired functions. Consider $(\mathscr{U}, \mathscr{U}_B), (\mathscr{V}, \mathscr{V}_B) \in \Gamma(Y, B)$ such that $(\mathscr{U}, \mathscr{U}_B) < (\mathscr{V}, \mathscr{V}_B)$. Using Lemma 3.5 and taking homology, we have the following diagram commutes

$$\begin{tikzcd}
H_n(X, A; \mathscr{V}', \mathscr{V}'_A) \arrow[r, "f_{\mathscr{V}*}"] \arrow[d, "\pi_{\mathscr{U}'\mathscr{V}'*}"'] & H_n(Y, B; \mathscr{V}, \mathscr{V}_B) \arrow[d, "\pi_{\mathscr{U}\mathscr{V}*}"] \\
H_n(X, A; \mathscr{U}', \mathscr{U}'_A) \arrow[r, "f_{\mathscr{U}*}"'] & H_n(Y, B; \mathscr{U}, \mathscr{U}_B)
\end{tikzcd}$$

It follows that $\{f_{\mathscr{U}*} : H_n(X, A; \mathscr{U}', \mathscr{U}'_A) \to H_n(Y, B; \mathscr{U}, \mathscr{U}_B), (\mathscr{U}, \mathscr{U}_B) \in \Gamma(Y, B)\}$ is an inverse system of homomorphisms. Thus, using Theorem 2.7, there exists a unique homomorphism

$$f_* : \check{H}_n(X, A) \to \check{H}_n(Y, B)$$

that satisfies 3.3.

Similarly by taking cohomology, we have that

$$\begin{tikzcd}
H^n(Y, B; \mathscr{U}, \mathscr{U}_B) \arrow[r, "f^*_{\mathscr{U}}"] \arrow[d, "\pi^*_{\mathscr{V}\mathscr{U}}"'] & H^n(X, A; \mathscr{U}', \mathscr{U}'_A) \arrow[d, "\pi^*_{\mathscr{V}'\mathscr{U}'}"] \\
H^n(Y, B; \mathscr{V}, \mathscr{V}_B) \arrow[r, "f^*_{\mathscr{V}}"'] & H^n(X, A; \mathscr{V}', \mathscr{V}'_A)
\end{tikzcd}$$

and so $\{f^*_{\mathscr{U}} : H^n(Y, B; \mathscr{U}, \mathscr{U}_B) \to H^n(X, A; \mathscr{U}', \mathscr{U}'_A), (\mathscr{U}, \mathscr{U}_B) \in \Gamma(Y, B)\}$ is a direct system of homomorphisms. Therefore, using Theorem 2.16, there exists a unique homomorphism

$$f^* : \check{H}^*(Y, B) \to \check{H}^*(X, A)$$

that satisfies 3.4. □

With this last Theorem, we will show that the Čech Homology and Cohomology we defined are functors, since the induced homomorphisms satisfy the functorial properties.

Theorem 3.7. *Let $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ be continuous functions. Then, using the corresponding induced homomorphisms defined in Theorem 3.6 satisfy that*

$$(gf)_* = g_* f_* \quad \textit{and} \quad (gf)^* = f^* g^*$$

Furthermore,

$$(Id_X)_* = Id_{\check{H}_*(X,A)} \quad \textit{and} \quad (Id_X)^* = Id_{\check{H}^*(X,A)}$$

Proof. Recall that $\{H_n(X, A; (\mathscr{U}, \mathscr{U}_A)), (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)\}$, $\{H_n(Y, B; (\mathscr{U}, \mathscr{U}_B)), (\mathscr{U}, \mathscr{U}_B) \in \Gamma(Y, B)\}$ and $\{H_n(Z, C; (\mathscr{W}, \mathscr{W}_C)), (\mathscr{W}, \mathscr{W}_C) \in \Gamma(Z, C)\}$ are inverse systems, and both $g^{-1} : \Gamma(Z, C) \to \Gamma(Y, B)$ and $f^{-1} : \Gamma(Y, B) \to \Gamma(X, C)$ are order preserving maps. Also, using Lemma 3.5, we have that both $\{f_{\mathscr{U}*} : H_n(X, A; \mathscr{U}', \mathscr{U}'_A) \to H_n(Y, B; \mathscr{U}, \mathscr{U}_B), (\mathscr{U}, \mathscr{U}_B) \in \Gamma(Y, B)\}$ and $\{g_{\mathscr{U}*} : H_n(Y, B; \mathscr{W}', \mathscr{W}'_B) \to H_n(Z, C; \mathscr{W}, \mathscr{W}_C), (\mathscr{W}, \mathscr{W}_C) \in \Gamma(Z, C)\}$ are inverse systems of homomorphisms. Thus, using Theorem 2.8, we have that in fact

$$(gf)_* = g_* f_* .$$

The case for cohomology is similar, since the Lemma 3.5 is on the simplicial maps and using Theorem 2.17, we have that

$$(gf)^* = f^* g^* .$$

Finally, using Observation 8 and by taking homology, we have that for all $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$ the induced maps $(\mathrm{Id}_X)_{\mathscr{U}*}$ and $(\mathrm{Id}_X)^*_{\mathscr{U}}$ are the corresponding identities. Therefore, using Theorem 3.6, we have that $(\mathrm{Id}_X)_* = \mathrm{Id}_{\check{H}_*(X,A)}$ and $(\mathrm{Id}_X)^* = \mathrm{Id}_{\check{H}^*(X,A)}$. □

3.2 Homotopy invariance

Theorem 3.8. *If $f, g : (X, A) \to (Y, B)$ are homotopic maps in **Cl**, then the induced maps on homology and cohomology are the same.*

3.2.1 Proof of Theorem 3.8

Lemma 3.9. *Let $\mathscr{V} \in \Gamma(I)$ be a finite open cover of connected sets. Then $K_{\mathscr{V}}$ is acyclic (recall definition 2.7).*

Proof. We will suppose there is no inclusions between different sets of the cover $\mathscr{V}$. If $V_1, V_2 \in \mathscr{V}$ are such that $V_1 \subset V_2$. Let V' be the cover $\mathscr{V}$ without V_1. Then $\mathscr{V}' < \mathscr{V}$, since $\mathscr{V}'$ is a subcollection of $\mathscr{V}$, and $\mathscr{V} < \mathscr{V}'$ because $V_2 \in \mathscr{V}'$ and $V_1 \subset V_2$. Thus, $K_{\mathscr{V}}$ and $K_{\mathscr{V}'}$ are isomorphic on (co)homology, and so, in order to prove the lemma we will focus on covers such that no inclusions between different sets.

Now, with the hypothesis we set before, we can take $\mathscr{V} = \{V_0, \ldots, V_n\}$ such that $V_j = (a_j, b_j)$, and that $a_j < a_{j+1}$ and $b_j < b_{j+1}$, with $a_0 = 0$ and $b_n = 1$. For each $i = 0, \ldots, n$, consider the simplicial maps $f_i : K_{\mathscr{V}} \to K_{\mathscr{V}}$ defined on the vertices by

$$f_i(V_j) = \begin{cases} V_j & ,\text{for } j \leq i \\ V_i & ,\text{for } j > i \end{cases}$$

We will show that f_i and f_{i+1} are contiguous. Let S be a simplex of $K_{\mathscr{V}}$. If the indices of the vertices of S are less or equal than i, then $f_i(S) = f_{i+1}(S)$. If some of the indices of the vertices of S are more than i, then V_{i+1} is a vertex of $f_{i+1}(S)$, and there are two possibilities: V_i is a vertex of $f_{i+1}(S)$ or it isn't a vertex. In the first case, we have that $f_{i+1}(S)$ has all the vertices of $f_i(S)$. In the second case, let $\{V_{j_0}, \ldots, V_{j_k}\}$ be the vertices of S such that $j_l < i$, for $l = 0, \ldots, k$. Since each element of $\mathscr{V}$ are connected, there are $a, b \in I$ such that $a < a_i,\ b < b_i$, and

$$\bigcap_{l=0}^{k} V_{j_l} = (a, b)$$

Using that $f_{i+1}(S)$ is a simplex, we have that $(a, b) \cap V_{i+1} \neq \emptyset$, and so $a < a_i < a_{i+1} < b < b_i$. Thus, $(a, b) \cap V_i \cap V_{i+1} \neq \emptyset$. It follows that $\{V_{j_0}, \ldots, V_{j_k}, V_i, V_{i+1}\}$ are vertices of a simplex in $K_{\mathscr{V}}$. Therefore, f_{i+1} and f_i are contiguous.

Finally, note that f_n is the identity map and that f_0 is a constant map. Since $f_{n*} = f_{0*}$ and ${f_n}^* = {f_0}^*$, we conclude that $K_{\mathscr{V}}$ is acyclic. □

Definition 3.5. A finite cover $\mathscr{V} = \{V_0, \ldots, V_n\}$ of I it's called *regular* if each one of the elements is open and connected, and if we can index the sets such that

- $V_i \cap V_{i+1} \neq \emptyset \quad \text{for} \quad i = 0, \ldots, n-1$
- $V_i \cap V_j = \emptyset \quad \text{for} \quad j < i-1$, for $i = 1, \ldots, n$
- $0 \in V_0, \quad 1 \in V_n$, and $0 \notin V_1, 1 \notin V_{n-1}$.

Lemma 3.10. *The set of all regular covers of I is a cofinal subset $\Gamma(I)$, the set of all interior covers of I.*

Proof. Let $\mathscr{V} \in \Gamma(I)$. Since I is a topological space, we have that $i_I(i_I(A)) = i_I(A)$. It follows that $\mathscr{V}' = \{i_I(V) \,|V \in \mathscr{V}\}$ is also an interior cover of I, which is also a refinement of $\mathscr{V}$, because $i_I(V) \subset V$ for all $V \subset I$.

Using that I is compact and that $\mathscr{V}'$ is an open cover of I, we can consider there is a finite refinement of open intervals. The result will be proved with induction. Suppose that there are different intervals $\{U_0, \ldots, U_k\}$, ordered from left to right, with $U_i \cap U_j = \emptyset$, except when $j = i+1$, such that each U_j is contain in an element of $\mathscr{V}'$. If $1 \neq U_k$, then we have that $U_j = (a_j, b_j)$, for each $0 < j \leq k$, with $U_0 = [0, b_0)$. Thus, there is $V_{k+1} \in \mathscr{V}'$ such that $b_k \in V_{k+1}$. Since V_{k+1} is open, there is an open interval $U_{k+1} := (a_{k+1}, b_{k+1}) \subset V_{k+1}$, with $b_{k-1} < a_{k+1} < b_k < b_{k+1}$, and so $U_j \cap U_{k+1} = \emptyset$, except with $j = k$. Since $\{b_j\}$ is an increasing sequence, we can cover I with a finite number of term, and the cover $\{U_0, \ldots, U_n\}$ is a regular cover. □

Definition 3.6. Let $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$ be a cover. Suppose for each $U \in \mathscr{U}$ there is a regular cover $\{V_{U,0}, \ldots, V_{U,n_U}\} =: \mathscr{V}_U \in \Gamma(I)$. Consider the interior cover $(\mathscr{W}, \mathscr{W}_{A\times I})$ of $(X \times I, A \times I)$

defined by

$$\mathscr{W} := \{U \times V \mid U \in \mathscr{U}, V \in \mathscr{V}_U\} \quad \text{and} \quad \mathscr{W}_A := \{U \times V \mid U \in \mathscr{U}_A, V \in \mathscr{V}_U\}$$

We will call this an *interior cover of* $X \times I$ *stacked over* $(\mathscr{U}, \mathscr{U}_A)$. Also, we will refer to $\mathscr{V}_U$ as *the stack corresponding at* U.

For convenience, we will write $U \times V_{U,i} \in \mathscr{W}$ by (U, i), for each $U \in \mathscr{U}$ and $i \in \{0, \ldots, n_U\}$.

Lemma 3.11. *The subset of stacked covers is cofinal in* $\Gamma(X \times I, A \times I)$*.*

Proof. First we are going to reference some tools we have shown before. Recall from Theorem 1.6 that for any local base $\mathcal{B}_{(x,t)}$, if $W \subset X \times I$, then we have that $(x, t) \in i_{X,I}(W)$ if and only if there is $B \in \mathcal{B}_{(x,t)}$ such that $B \subset W$. Also recall from the Definition 1.8 that for any $(x, t) \in X \times I$,

$$\mathcal{B}_{(x,t)} = \{U \times V |\, U \in \mathcal{N}_x,\ V \in \mathcal{N}_t\}$$

is a local base at (x, t), where $\mathcal{N}_x,\ \mathcal{N}_t$ are the neighborhood systems of $x \in X$ and $t \in I$. Finally, from Proposition 1.10, for any $U \subset X$ and $V \subset I$, we have that $i_{X,I}(U \times V) = i_X(U) \times i_I(V)$.

Let $(\mathscr{W}, \mathscr{W}_{A\times I}) \in \Gamma(X \times I, A \times I)$. Fix $x \in X$, then we have that for each $t \in I$, there is $W \in \mathscr{W}$ such that $(x, t) \in i_{X,I}(W)$, since $\mathscr{W}$ is an interior cover. Thus, there are $U_{x,t} \in \mathcal{N}_x$ and $V_{x,t} \in \mathcal{N}_t$ such that $U_{x,t} \times V_{x,t} \subset W$. Let $\mathscr{V}'_x$ be the collection of all the sets of the form $V_{x,t}$, with $t \in I$. Then $\mathscr{V}'_x$ is an interior cover of I.

Using that regular covers are cofinal in $\Gamma(I)$, there exits a regular cover $\mathscr{V}_x = \{V_0, \ldots, V_{n_x}\}$ of I that is a refinement of $\mathscr{V}'_x$. It follows that for each $j \in \{0, \ldots, n_x\}$ there is $t_j \in I$ such that $V_j \subset V_{x,t_j}$. Also, for each t_j choose $W_j \in \mathscr{W}$ such that $(x, t_j) \in i_{X,I}(W_j)$, and $U_{x,t_j} \in \mathcal{N}_x$ such that $U_{x,t_j} \times V_j \subset U_{x,t_j} \times V_{x,t_j} \subset W_j$. We now define $U_x := \bigcap\limits_{j=0}^{n_x} U_{x,t_j} \in \mathcal{N}_x$, since the neighborhood system is a filter. It follows that $U_x \times V_j \subset W_j$, for each $j = 0, \ldots, n_x$. Also, if $x \in A$, we can suppose that $W_j \in \mathscr{W}_{A\times I}$.

Let $\mathscr{U}$ be the collection of all the sets U_x we defined above. We have that $\mathscr{U}$ is an interior cover of X, since each U_x is a neighborhood of x. For each $(x, t) \in X \times I$, there is $U_x \in \mathcal{N}_x$ and $V_j \in V_x$ such that $(x, t) \in i_X(U_x) \times i_I(V_j) = i_{X,I}(U_x \times V_j)$. Similarly, we have the same result for any $(x, t) \in A \times I$. Therefore, if we define

$$\mathscr{W}' := \{U_x \times V_j | x \in X, V_j \in \mathscr{V}_x\} \quad \text{and} \quad \mathscr{W}'_{A\times I} := \{U_x \times V_j | x \in A, V_j \in \mathscr{V}_x\},$$

we have that $(\mathscr{W}', \mathscr{W}'_{A\times I})$ an interior cover stacked over $(\mathscr{U}, \mathscr{U}_A)$, which is a refinement of $(\mathscr{W}, \mathscr{W}_{A\times I})$. □

Lemma 3.12. *Let* $(\mathscr{W}, \mathscr{W}_{A\times I}) \in \Gamma(X \times I, A \times I)$ *be a stacked covering over* $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$*. If the nerve* $K_{\mathscr{U}}$ *is a (finite) simplex, then the nerve* $K_{\mathscr{W}}$ *is acyclic.*

Proof. Without loss of generality, we suppose that $U \in \mathscr{U}$ implies $U \neq \emptyset$. For each $U \in \mathscr{U}$, let $\mathscr{V}_U$ be the corresponding stack. Define

$$\mathscr{V} = \{V \subset I | U \times V \in \mathscr{W}\}$$

and note that this forms a cover of I. Let s be a simplex of $K_{\mathscr{W}}$ with vertices $\{U_0 \times V_0, \ldots, U_n \times V_n\}$ in $\mathscr{W}$. Note that for each $i \in \{0, \ldots, n\}$ there is a $j_i \in \{0, \ldots, n_{U_i}\}$ such that $V_i = V_{U_i, j_i}$. Then

$$\begin{aligned} \bigcap_{i=0}^{n} (U_i \times V_i) &= \left(\bigcap_{i=0}^{n} U_i \right) \times \left(\bigcap_{i=0}^{n} V_i \right) \\ &= \left(\bigcap_{i=0}^{n} U_i \right) \times \left(\bigcap_{i=0}^{n} V_{U_i, j_i} \right). \end{aligned}$$

Since, by hypothesis, $K_{\mathscr{U}}$ is a simplex, we have that

$$\bigcap_{i=0}^{n} U_i \neq 0$$

Therefore, $(U_0 \times V_0) \cap \ldots \cap (U_n \times V_n) \neq \emptyset$ if and only if $V_0 \cap \ldots \cap V_n \neq \emptyset$. Thus $K_{\mathscr{W}} = K_{\mathscr{V}}$. Since $\mathscr{V}$ is a finite open cover of I by connected sets, its nerve is acyclic. $\square$

Lemma 3.13. *Let $(\mathscr{W}, \mathscr{W}_{A \times I}) \in \Gamma(X \times I, A \times I)$ be a covering stacked over the covering $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$. Consider the simplicial maps*

$$l, u : (K_{\mathscr{U}}, L_{\mathscr{U}}) \to (K_{\mathscr{W}}, L_{\mathscr{W}})$$

defined for $U \in \mathscr{U}$ by

$$l(U) = (U, 0), \quad \textit{and} \quad u(U) = (U, n_U)$$

Then, the induced maps on (co)homology are the same, i.e.,

$$l_* = u_* \quad \textit{and} \quad l^* = u^*$$

Proof. Given a simplex S of $K_{\mathscr{U}}$, consider the subcomplex $C(S)$ of $K_{\mathscr{W}}$ consisting of all simplexes whose vertices have the form (U, i) such that U is a vertex of S. Define

$$\mathscr{U}' := \{U \in \mathscr{U} | U \text{ vertex of } S\} \quad \text{and} \quad X' = \bigcup_{U \in \mathscr{U}'} U$$

Thus, we have that S is the nerve of the covering $\mathscr{U}'$ of X', and $C(S)$ is the nerve of a covering $\mathscr{W}'$ stacked over $\mathscr{U}'$. Using Lemma 3.12, $C(S)$ is acyclic. Thus, C is an acyclic carrier, and so, using 2.1, we have that $l_* = u_*$ and $l^* = u^*$.

$\square$

Theorem 3.14. *Let $g_0, g_1 : (X, A) \to (X \times I, A \times I)$ be defined by*

$$g_0(x) = (x, 0) \quad \textit{and} \quad g_1(x) = (x, 1)$$

Then the induced homomorphisms on (co)homology are the same, i.e.,

$$g_{0*} = g_{1*} \quad \textit{and} \quad g_{0*} = g_{1*}$$

Proof. Since the set of stacked coverings is cofinal, we will prove the result considering them. Let $(\mathscr{W}, \mathscr{W}_{A\times I}) \in \Gamma(X \times I, A \times I)$ be a stacked cover over $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$. Consider the covers of (X, A) given by $(\mathscr{U}_0, \mathscr{U}_{0A}) := {g_0}^{-1}(\mathscr{W}, \mathscr{W}_{A\times I})$ and $(\mathscr{U}_1, \mathscr{U}_{1A}) := {g_1}^{-1}(\mathscr{W}, \mathscr{W}_{A\times I})$. By definition of a stacked cover, we have that for any $U \in \mathscr{U}$ there is a regular cover $\mathscr{V}_U := V_{U,0}, \ldots, V_{U,n_U}$ of I. Recall, from the definition of a regular cover, that $0 \in V_{U,i}$ if and only if $i = 0$, and so $(x, 0) \in U \times V_{U,i}$ if and only if $x \in U$ and $i = 0$. Similarly, we have that $(x, 1) \in U \times V_{U,i}$ if and only if $x \in U$ and $i = n_U$. With this we can consider the maps $g_{j\mathscr{W}} : (K_{\mathscr{U}_j}, L_{\mathscr{U}_{jA}}) \to (K_{\mathscr{W}}, L_{\mathscr{W}})$ as inclusions, for j=0,1,, because they can be defined on the vertices by $g_{0\mathscr{W}}(U) = (U, 0)$ and $g_{1\mathscr{W}}(U) = (U, n_U)$. Thus we will consider $(K_{\mathscr{U}_0}, L_{\mathscr{U}_{0A}})$ and $(K_{\mathscr{U}_1}, L_{\mathscr{U}_{1A}})$ as subcomplexes of $(K_{\mathscr{W}}, L_{\mathscr{A}_{A\times I}})$. Now, consider the map $\pi_0 : (K_{\mathscr{W}}, L_{\mathscr{W}_{A\times I}}) \to (K_{\mathscr{U}_0}, L_{\mathscr{U}_{0A}})$ defined on the vertices by $\pi_0(U, i) = (U, 0)$. For convenience, we will refer to π_0 as the restriction corresponding to $(K_{\mathscr{U}_1}, L_{\mathscr{U}_{1A}})$. On the other hand, we have a simplicial map $\eta : (K_{\mathscr{U}}, L_{\mathscr{U}_A}) \to (K_{\mathscr{U}_1}, L_{\mathscr{U}_{1A}})$ defined on the vertices by $\eta(U) = (U, n_U)$. Since $\mathscr{W}$ is stacked, we have that $(\mathscr{U}_0, \mathscr{U}_{0A}) = (\mathscr{U}_1, \mathscr{U}_{1A}) = (\mathscr{U}, \mathscr{U}_A)$ as covers of X, and so the simplicial maps π_0, η, and $\eta\pi_0$ correspond to the simplicial maps induced by the refinement $\mathscr{U}_0 < \mathscr{U}_1$, $\mathscr{U}_1 < \mathscr{U}$, and $\mathscr{U}_0 < \mathscr{U}$. It follows that the maps u, l, defined in Lemma 3.13, can be written as $u = g_{1\mathscr{W}}\eta$ and $l = g_{0\mathscr{W}}\pi_0\eta$. Thus, we have the following commutative diagram on homology

$$\begin{tikzcd}
 & H_*(K_{\mathscr{U}_1}, L_{\mathscr{U}_{1A}}) \arrow[d, "\pi_0{}_*"] \arrow[dr, "g_{1\mathscr{W}*}"] & \\
H_*(K_{\mathscr{U}}, L_{\mathscr{U}_A}) \arrow[ur, "\eta_*"] \arrow[dr, dashed, "(\pi_0\eta)_*"'] \arrow[rr, bend left=60, "u_*"] \arrow[rr, bend right=60, "l_*"'] & & H_*(K_{\mathscr{W}}, L_{\mathscr{W}_{A\times I}}) \\
 & H_*(K_{\mathscr{U}_0}, L_{\mathscr{U}_{0A}}) \arrow[ur, "g_{0\mathscr{W}*}"'] &
\end{tikzcd}$$

since $u_* = l_*$. Using that η_* is an isomorphism, we have that $g_{1\mathscr{W}_*} = g_{0\mathscr{W}_*}\pi_0{}_*$. We also have the following commutative diagram, for $j = 0, 1$,

$$
\begin{tikzcd}
\check{H}_*(X, A) \arrow[r, "{g_j}_*"] \arrow[d, "\pi_{\mathscr{U}_j}"'] & \check{H}_*(X \times I, A \times I) \arrow[d, "\kappa_{\mathscr{W}}"] \\
H_*(K_{\mathscr{U}_j}, L_{\mathscr{U}_{jA}}) \arrow[r, "{g_j}_{\mathscr{W}_*}"'] & H_*(K_{\mathscr{W}}, L_{\mathscr{W}_A})
\end{tikzcd}
$$

where $\pi_{\mathscr{U}}$ and $\kappa_{\mathscr{W}}$ are the natural projections from the respective inverse limits. As we stated before, π_0 correspond to the simplicial maps induced by the refinement $\mathscr{U}_0 < \mathscr{U}_1$, and so ${\pi_0}_* \pi_{\mathscr{U}_1} = \pi_{\mathscr{U}_0}$. It follows that

$$
\kappa_{\mathscr{W}} {g_1}_* = {g_1}_{\mathscr{W}_*} \pi_{\mathscr{U}_1} = {g_0}_{\mathscr{W}_*} {\pi_0}_* \pi_{\mathscr{U}_1} = {g_0}_{\mathscr{W}_*} \pi_{\mathscr{U}_0} = \kappa_{\mathscr{W}} {g_0}_*
$$

By uniqueness of the inverse limit of homomorphisms, we conclude that ${g_1}_* = {g_0}_*$.

Similarly, on cohomology we have the following commutative diagram

$$
\begin{tikzcd}
 & H^*(K_{\mathscr{U}_1}, L_{\mathscr{U}_1}) \arrow[dr, "\eta^*"] & \\
H^*(K_{\mathscr{W}}, L_{\mathscr{W}}) \arrow[ur, "{g_1}^*_{\mathscr{W}}"] \arrow[dr, "{g_0}^*_{\mathscr{W}}"'] \arrow[rr, bend left=60, "u^*"] \arrow[rr, bend right=60, "l^*"'] & & H^*(K_{\mathscr{U}}, L_{\mathscr{U}}) \\
 & H_*(K_{\mathscr{U}_0}, L_{\mathscr{U}_0}) \arrow[uu, "{\pi_0}^*"] \arrow[ur, dashed, "(\pi_0\eta)^*"'] &
\end{tikzcd}
$$

from which we have that ${g_1}^*_{\mathscr{W}} = {\pi_0}^* {g_0}^*_{\mathscr{W}}$, since $u^* = l^*$ and η^* is an isomorphism. For $j = 0, 1$, we have the following commutative diagram

$$
\begin{tikzcd}
H_*(K_{\mathscr{W}}, L_{\mathscr{W}_{A \times I}}) \arrow[r, "{g_j}^*_{\mathscr{W}}"] \arrow[d, "\kappa^{\mathscr{W}}"'] & H_*(K_{\mathscr{U}}, L_{\mathscr{U}_A}) \arrow[d, "\pi^{\mathscr{U}_j}"] \\
\check{H}_*(X \times I, A \times I) \arrow[r, "{g_j}^*"'] & \check{H}_*(X, A)
\end{tikzcd}
$$

where $\pi^{\mathscr{U}_j}$ and $\kappa^{\mathscr{W}}$ are the natural inclusions from the direct limit. It follows that

$$
{g_1}^* \kappa^{\mathscr{W}} = \pi^{\mathscr{U}_1} {g_1}^*_{\mathscr{W}} = \pi^{\mathscr{U}_1} {\pi_0}^* {g_0}^*_{\mathscr{W}} = \pi^{\mathscr{U}_0} {g_0}^*_{\mathscr{W}} = {g_0}^* \kappa^{\mathscr{W}}
$$

Thus, by uniqueness of the direct limit of homomorphisms, we have that ${g_1}^* = {g_0}^*$. $\square$

3.3 Exactness axiom

Theorem 3.15. *Consider a pair (X, A) with inclusions $\iota : A \to X$ and $j : X \to (X, A)$. Then, for each $n > 1$, there exists homomorphisms*

$$\partial_n : H_n(X, A) \to H_{n-1}(A) \quad \textit{and} \quad \delta_n : H^n(A) \to H^{n+1}(X, A)$$

such that the sequence on homology

$$\ldots \to H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \ldots$$

is a partial sequence, and the sequence on cohomology

$$\ldots \leftarrow H^n(A) \xleftarrow{\iota_*} H^n(X) \xleftarrow{j_*} H^n(X, A) \xleftarrow{\delta} H^{n-1}(A) \leftarrow \ldots$$

is exact. Furthermore, if $f : (X, A) \to (Y, B)$ is a continuous function, then the following diagrams commute

$$\begin{tikzcd} H_n(X,A) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{n-1}(A) \arrow[d, "f_*"] \\ H_n(Y,B) \arrow[r, "\partial"'] & H_{n-1}(B) \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} H^n(A) \arrow[r, "\delta"] \arrow[d, "f^*"'] & H^{n+1}(X,A) \arrow[d, "f^*"] \\ H^n(B) \arrow[r, "\delta"'] & H^{n+1}(Y,B) \end{tikzcd}$$

3.3.1 Proof of Theorem 3.15

Let $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)$ be a covering of (X, A). For the simplicial pair $(K_{\mathscr{U}}, L_{\mathscr{U}_A})$, along with the inclusion maps $j_{\mathscr{U}} : (K_{\mathscr{U}}, \emptyset) \to (K_{\mathscr{U}}, L_{\mathscr{U}_A})$ and $h_{\mathscr{U}} : (L_{\mathscr{U}_A}, \emptyset) \to (K_{\mathscr{U}}, \emptyset)$, there is an homology exact sequence

$$\cdots \to H_n(L_{\mathscr{U}_A}) \xrightarrow{h_{\mathscr{U}*}} H_n(K_{\mathscr{U}}) \xrightarrow{j_{\mathscr{U}*}} H_n(K_{\mathscr{U}}, L_{\mathscr{U}_A}) \xrightarrow{\partial'_{\mathscr{U}}} H_{n-1}(L_{\mathscr{U}_A}) \to \cdots \tag{3.5}$$

and a cohomology exact sequence

$$\cdots \leftarrow H^n(L_{\mathscr{U}}) \xleftarrow{h^*_{\mathscr{U}}} H^n(K_{\mathscr{U}}) \xleftarrow{j^*_{\mathscr{U}}} H^n(K_{\mathscr{U}}, L_{\mathscr{U}}) \xleftarrow{\delta'_{\mathscr{U}}} H^{n-1}(L_{\mathscr{U}}) \leftarrow \cdots \tag{3.6}$$

Now, we will prove that $\{h_{\mathscr{U}*},\ (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)\}$, $\{j_{\mathscr{U}*},\ (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)\}$, and $\{\partial'_{\mathscr{U}},\ (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)\}$ are inverse systems; and that $\{h^*_{\mathscr{U}},\ (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)\}$, $\{j^*_{\mathscr{U}},\ (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)\}$, and $\{\delta'_{\mathscr{U}},\ (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)\}$ are direct systems

Let $(\mathscr{V}, \mathscr{V}_A) \in \Gamma(X, A)$ such that $(\mathscr{U}, \mathscr{U}_A) < (\mathscr{V}, \mathscr{V}_A)$. First, we have a simplicial map $\pi^1_{\mathscr{U}\mathscr{V}} : (K_{\mathscr{V}}, L_{\mathscr{V}_A}) \to (K_{\mathscr{U}}, L_{\mathscr{U}_A})$. Since $\pi^1_{\mathscr{U}\mathscr{V}}(L_{\mathscr{V}_A}) \subset L_{\mathscr{U}_A}$, we have that the following commutative diagram

$$\begin{array}{ccc} L_{\mathscr{V}_A} & \xrightarrow{h_{\mathscr{V}}} & K_{\mathscr{V}} \\ {\scriptstyle \pi^1_{\mathscr{U}\mathscr{V}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi^1_{\mathscr{U}\mathscr{V}}} \\ L_{\mathscr{U}_A} & \xrightarrow[h_{\mathscr{U}}]{} & K_{\mathscr{U}} \end{array}$$

and so, on homology $\{h_{\mathscr{U}*},\ (\mathscr{U},\mathscr{U}_A) \in \Gamma(X,A)\}$ is an inverse system and on cohomology $\{h^*_{\mathscr{U}},\ (\mathscr{U},\mathscr{U}_A) \in \Gamma(X,A)\}$ is an direct system. Similarly, we have the following commutative diagram

$$\begin{array}{ccc} K_{\mathscr{V}} & \xrightarrow{j_{\mathscr{V}}} & (K_{\mathscr{V}}, L_{\mathscr{V}_A}) \\ {\scriptstyle \pi^1_{\mathscr{U}\mathscr{V}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi^1_{\mathscr{U}\mathscr{V}}} \\ K_{\mathscr{U}} & \xrightarrow[j_{\mathscr{U}}]{} & (K_{\mathscr{U}}, L_{\mathscr{U}_A}) \end{array}$$

Thus, we have that $\{j_{\mathscr{U}*},\ (\mathscr{U},\mathscr{U}_A) \in \Gamma(X,A)\}$ and $\{j^*_{\mathscr{U}},\ (\mathscr{U},\mathscr{U}_A) \in \Gamma(X,A)\}$ are inverse and direct systems, respectively. Using the properties of the exactness axiom of simplicial homology, we have that the following commutative diagram

$$\begin{array}{ccc} H_n(K_{\mathscr{V}}, L_{\mathscr{V}_A}) & \xrightarrow{\partial'_{\mathscr{V}}} & H_{n-1}(L_{\mathscr{V}_A}) \\ {\scriptstyle \pi_{\mathscr{U}\mathscr{V}*}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathscr{U}\mathscr{V}*}} \\ H_n(K_{\mathscr{U}}, L_{\mathscr{U}_A}) & \xrightarrow[\partial'_{\mathscr{U}}]{} & H_{n-1}(L_{\mathscr{U}_A}) \end{array}$$

We conclude that $\{\partial'_{\mathscr{U}},\ (\mathscr{U},\mathscr{U}_A) \in \Gamma(X,A)\}$ is an inverse system. Similarly, from the exactness axiom of simplicial cohomology, we have that the following diagram commutes

$$\begin{array}{ccc} H^n(L_{\mathscr{U}_A}) & \xrightarrow{\delta'_{\mathscr{U}}} & H^{n+1}(K_{\mathscr{U}}, L_{\mathscr{U}_A}) \\ {\scriptstyle \pi^*_{\mathscr{U}\mathscr{V}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi^*_{\mathscr{U}\mathscr{V}}} \\ H^n(L_{\mathscr{V}_A}) & \xrightarrow[\delta'_{\mathscr{V}}]{} & H^{n+1}(K_{\mathscr{V}}, L_{\mathscr{V}_A}) \end{array}$$

Therefore, $\{\delta'_{\mathscr{U}},\ (\mathscr{U},\mathscr{U}_A) \in \Gamma(X,A)\}$ is an direct system.

If we write the subscript corresponding to the direct set $\Gamma(X,A)$ whenever the limit process differs from the direct set we defined the respective Čech homology for the groups and homomorphisms on the sequence (3.5), we have the following sequence

$$\ldots \to \check{H}_n\,(A)_{\Gamma(X,A)} \xrightarrow{h_{*\Gamma(X,A)}} \check{H}_n\,(X)_{\Gamma(X,A)} \xrightarrow{j_{*\Gamma(X,A)}} \check{H}_n\,(X,A) \xrightarrow{\partial'_*} \check{H}_{n-1}\,(A)_{\Gamma(X,A)} \to \ldots \tag{3.7}$$

which is of order two, since for each $(\mathscr{U},\mathscr{U}_A) \in \Gamma(X,A)$ the maps $\partial'_{\mathscr{U}} j_{\mathscr{U}*} = j_{\mathscr{U}*} h_{\mathscr{U}*} = h_{\mathscr{U}*} \partial'_{\mathscr{U}} = 0$. Similarly, we are going to write the subscript after taking direct limits of the groups and homomorphisms in the sequence (3.6), and we obtain the following sequence

$$\ldots \leftarrow \check{H}^n(A)_{\Gamma(X,A)} \xleftarrow{h^*{}_{\Gamma(X,A)}} \check{H}^n(X)_{\Gamma(X,A)} \xleftarrow{j^*{}_{\Gamma(X,A)}} \check{H}^n(X,A) \xleftarrow{\delta'^*} \check{H}^{n-1}(A)_{\Gamma(X,A)} \leftarrow \ldots \tag{3.8}$$

which is exact by using the Theorem 2.18.

Now, we want to relate $\check{H}_n(A)_{\Gamma(X,A)}$, $\check{H}_n(X)_{\Gamma(X,A)}$, $\check{H}^n(A)_{\Gamma(X,A)}$, and $\check{H}^n(X)_{\Gamma(X,A)}$, with the respective groups without the subscript. In order to achieve it, we define two maps

$$\begin{aligned} \psi : \Gamma(X,A) &\to \Gamma(X) \\ (\mathscr{U}, \mathscr{U}_A) &\mapsto \mathscr{U} \end{aligned}$$

and

$$\begin{aligned} \phi : \Gamma(X,A) &\to \Gamma(A) \\ (\mathscr{U}, \mathscr{U}_A) &\mapsto \iota^{-1}(\mathscr{U}_A) \end{aligned}$$

It follows from the definition of interior cover of a pair that ψ is well defined and that is an order preserving map. Also note that ψ is surjective, since for any $\mathscr{U} \in \Gamma(X)$, the pair $(\mathscr{U}, \mathscr{U}) \in \Gamma(X,A)$ is such that $\psi(\mathscr{U}, \mathscr{U}) = \mathscr{U}$. Note that for each $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X,A)$, we have that $(\mathrm{Id}_X)_{\mathscr{U}} : K_{\mathscr{U}} \to K_{\mathscr{U}}$, and so

$$\{(\mathrm{Id}_X)_{\mathscr{U}*} : H_n(K_{\mathscr{U}}) \to H_n(K_{\mathscr{U}}), (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X,A)\}$$

is an inverse system of isomorphisms of the system $\{H_n(K_{\mathscr{U}}), \pi_{\mathscr{U}\mathscr{V}*}, \Gamma(X)\}$ into $\{H_n(K_{\mathscr{U}}), \pi_{\mathscr{U}\mathscr{V}*}, \Gamma(X,A)\}$ corresponding to the order preserving map ψ. Similarly, we have that $\{(\mathrm{Id}_X)^*_{\mathscr{U}} : H^n(K_{\mathscr{U}}) \to H^n(K_{\mathscr{U}}), (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X,A)\}$ is an direct system of isomorphisms of the system $\{H^n(K_{\mathscr{U}}), \Gamma(X,A)\}$ into $\{H^n(K_{\mathscr{U}}), \Gamma(X)\}$ corresponding to the order preserving map ψ. Thus, we conclude that $\mathrm{Id}_{X*} : \check{H}_n(X) \to \check{H}_n(X)_{\Gamma(X,A)}$ and that $\mathrm{Id}_X{}^* : \check{H}^n(X)_{\Gamma(X,A)} \to \check{H}^n(X)$ are isomorphisms.

Now, consider the inclusion $\theta : (A,A) \to (X,A)$. Let $(\mathscr{U}, \mathscr{U}_A) \in \Gamma(X,A)$, and write $(\mathscr{U}', \mathscr{U}'_A) := \theta^{-1}(\mathscr{U}, \mathscr{U}_A)$. From Proposition 3.2, we have that $(\mathscr{U}', \mathscr{U}'_A) \in \Gamma(A,A)$, and from the definition of the interior cover, we also have that $\mathscr{U}'_A$ is an interior cover of A. For convenience, write $\mathscr{V} := \mathscr{U}'_A = \theta^{-1}(\mathscr{U}_A) = \iota^{-1}(\mathscr{U}_A)$. Consider the simplicial pair $(K_{\mathscr{U}'}, L_{\mathscr{U}'_A})$ corresponding to $(\mathscr{U}', \mathscr{U}'_A)$. From the definition of the subcomplex $L_{\mathscr{U}'_A}$, we have that $K_{\mathscr{V}} = L_{\mathscr{U}'_A}$. Now, using the simplicial map $\theta^1_{\mathscr{U}} : (K_{\mathscr{U}'}, L_{\mathscr{U}'_A}) \to (K_{\mathscr{U}}, L_{\mathscr{U}_A})$, we have that $\theta^1_{\mathscr{U}}(K_{\mathscr{V}}) = \theta^1_{\mathscr{U}}(L_{\mathscr{U}'_A}) \subset L_{\mathscr{U}_A}$, and so, we can factor the simplicial map $\theta^1_{\mathscr{U}}$ as

$$\begin{array}{ccc} L_{\mathscr{U}_A} & \xrightarrow{h_{\mathscr{U}}} & K_{\mathscr{U}} \\ \ell^1_{\mathscr{U}} \uparrow & \nearrow_{\theta^1_{\mathscr{U}}} & \\ K_{\mathscr{V}} & & \end{array}$$

where $\ell^1_{\mathscr{U}}$ is the restriction of the codomain of the simplicial map $\theta^1_{\mathscr{U}}$, and $h_{\mathscr{U}}$ is the inclusion from the sumcomplex $L_{\mathscr{U}_A}$ to $K_{\mathscr{U}}$.

Now, define the map $k_{\mathscr{U}} : L_{\mathscr{U}_A} \to K_{\mathscr{V}}$ by $k_{\mathscr{U}}(U) := U \cap A$, for each vertex U in $L_{\mathscr{U}_A}$. We will prove that $k_{\mathscr{U}}$ is a simplicial map. Let $U_0, \ldots, U_n$ be vertices of a simplex in $L_{\mathscr{U}_A}$, then

$$\emptyset \neq U_0 \cap \ldots \cap U_n \cap A = (U_0 \cap A), \ldots, (U_n \cap A) = k_{\mathscr{U}}(U_0), \ldots, k_{\mathscr{U}}(U_n)$$

Thus, we have that $k_{\mathscr{U}}(U_0), \ldots, k_{\mathscr{U}}(U_n)$ are vertices of a simplex of $K_{\mathscr{V}}$, and so $k_{\mathscr{U}}$ can be extended to a simplicial map as desired.

Note that $k_{\mathscr{U}}\ell^1_{\mathscr{U}} = \mathrm{Id}_{K_{\mathscr{V}}}$, because for any vertex V in $K_{\mathscr{V}}$ there is a vertex $U = \ell^1_{\mathscr{U}}(V)$ in $L_{\mathscr{U}_A}$ such that

$$V = \theta^{-1}(U) = U \cap A = k_{\mathscr{U}}(U) = k_{\mathscr{U}}(\ell^1_{\mathscr{U}}(V))$$

We also have that $\ell^1_{\mathscr{U}}k_{\mathscr{U}}$ and $\mathrm{Id}_{L_{\mathscr{U}_A}}$ are contiguous. In order to prove this, let $U_0, \ldots, U_n$ be vertices of a simplex in $L_{\mathscr{U}_A}$. If $U'_0, \ldots, U'_n$ are the respective images under $\ell^1_{\mathscr{U}}k_{\mathscr{U}}$. Then, for each $j = 0, \ldots, n$, we have that

$$U'_j \cap A = k_{\mathscr{U}}(U'_j) = k_{\mathscr{U}}(\ell^1_{\mathscr{U}}k_{\mathscr{U}}(U_j)) = k_{\mathscr{U}}(U_j) = U_j \cap A$$

and so,

$$\begin{aligned} U_0 \cap \ldots \cap U_n \cap U'_0 \cap \ldots \cap U'_n \cap A &= (U_0 \cap A) \cap \ldots \cap (U_n \cap A) \cap (U'_0 \cap A) \cap \ldots \cap (U'_n \cap A) \\ &= (U_0 \cap A) \cap \ldots \cap (U_n \cap A) \neq \emptyset \end{aligned}$$

This means that $U_0, \ldots, U_n, U'_0, \ldots, U'_n$ are vertices of a simplex of $L_{\mathscr{U}_A}$. Therefore, we conclude that $\ell^1_{\mathscr{U}}k_{\mathscr{U}}$ and the identity map are contiguous.

Now, we have that $\{\ell_{\mathscr{U}*} : H_n(K_{\mathscr{V}}) \to H_n(L_{\mathscr{U}_A}), (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)\}$ is an inverse system of isomorphisms of the system $\{H_n(K_{\mathscr{U}}), \Gamma(A)\}$ into $\{H_n(L_{\mathscr{U}_A}), \Gamma(X, A)\}$ corresponding to the order preserving map ϕ. Similarly, we have that $\{\ell^*_{\mathscr{U}} : H^n(L_{\mathscr{U}_A}) \to H^n(K_{\mathscr{V}}), (\mathscr{U}, \mathscr{U}_A) \in \Gamma(X, A)\}$ is an direct system of isomorphisms of the system $\{H^n(L_{\mathscr{U}_A}), \Gamma(X, A)\}$ into $\{H^n(K_{\mathscr{U}}), \Gamma(A)\}$ corresponding to the order preserving map ϕ.

Now, we show that ϕ is surjective. Let $\mathscr{V} \in \Gamma(A)$. Define

$$\mathscr{U} := \{U \subset X | U = X \text{ or } U = V \cup (X \setminus A),\ V \in \mathscr{V}\} \quad \text{and} \quad \mathscr{U}_A := \{V \cup (X \setminus A) | V \in \mathscr{V}\}.$$

Using Proposition 1.12, we have that

$$\begin{aligned} A &= \bigcup_{V \in \mathscr{V}} i_A(V) \\ &= \bigcup_{V \in \mathscr{V}} (i_X(V \cup (X \setminus A)) \cap A) \\ &\subset \bigcup_{V \in \mathscr{V}} i_X(V \cup (X \setminus A)) \\ &= \bigcup_{U \in \mathscr{U}_A} i_X(U) \end{aligned}$$

and so, $(\mathscr{U}, \mathscr{U}_A)$ is in fact an interior cover of the pair (X, A). Since for any $U \in \mathscr{U}_A$ there is a $V \in \mathscr{V}$ such that $U = V \cup (X \setminus A)$, we have that

$$\iota^{-1}(U) = U \cap A = (V \cup (X \setminus A)) \cap A = (V \cap A) \cup ((X \setminus A) \cap A) = V$$

Thus, we have that $\phi(\mathscr{U}, \mathscr{U}_A) = \mathscr{V}$. Now, we can say that there are isomorphisms

$$\ell_* : \check{H}_n(A) \to \check{H}_n(A)_{\Gamma(X,A)} \quad \text{and} \quad \ell^* : \check{H}^n(A)_{\Gamma(X,A)} \to \check{H}^n(A)$$

Now, by attaching the isomorphisms we defined before on the sequences (3.7) and (3.8), we obtain the following commutative diagrams

$$\begin{tikzcd} & \check{H}_n(A)_{\Gamma(X,A)} \arrow[r, "h_{*\Gamma(X,A)}"] & \check{H}_n(X)_{\Gamma(X,A)} \arrow[dr, "j_{*\Gamma(X,A)}"] & \\ \check{H}_{n+1}(X,A) \arrow[ur, "\partial'_*"] \arrow[dr, "\partial"'] & & & \check{H}_{n-1}(X,A) \\ & \check{H}_n(A) \arrow[uu, "\ell_*"] \arrow[r, "h_*"'] & \check{H}_n(X) \arrow[uu, "{\mathrm{Id}_X}_*"] \arrow[ur, "j_*"'] & \end{tikzcd}$$

where $\partial := (\ell_*)^{-1}\partial'_*$, and

$$\begin{tikzcd}
 & \check{H}^n(A)_{\Gamma(X,A)} \arrow[dl, "\delta'^*"'] \arrow[dd, "\ell^*"] & \check{H}^n(X)_{\Gamma(X,A)} \arrow[l, "h^*_{\Gamma(X,A)}"'] \arrow[dd, "\mathrm{Id}_X{}^*"] & \\
\check{H}^{n+1}(X,A) & & & \check{H}^{n-1}(X,A) \arrow[ul, "j^*_{\Gamma(X,A)}"'] \arrow[dl, "j^*"] \\
 & \check{H}^n(A) \arrow[ul, "\delta"] & \check{H}^n(X) \arrow[l, "h^*"] &
\end{tikzcd}$$

where $\delta := \delta'^*(\ell^*)^{-1}$. Recall that the sequence (3.8) is exact, using Proposition 5.1, we have that the bottom sequence is exact.

Finally, we need to prove that the homomorphisms ∂ and δ satisfy functorial properties. Let $f : (X,A) \to (Y,B)$ be a continuous function. If $g : A \to B$ is the restriction of f on the domain and codomain, we have the following commutative diagram on homology

$$\begin{tikzcd}
\check{H}_{n+1}(X,A) \arrow[r, "\partial'_*"] \arrow[d, "f_*"'] \arrow[rr, bend left, "\partial"] & \check{H}_n(A)_{\Gamma(X,A)} \arrow[d, "g_{*\Gamma(X,A)}"'] & \check{H}_n(A) \arrow[l, "\ell_*"'] \arrow[d, "g_*"] \\
\check{H}_{n+1}(Y,B) \arrow[r, "\partial'_*"'] \arrow[rr, bend right, "\partial"'] & \check{H}_n(B)_{\Gamma(Y,B)} & \check{H}_n(B) \arrow[l, "\ell_*"]
\end{tikzcd}$$

It follows that $\partial f_* = g_*\partial$. Similarly on cohomology, we have the commutative diagram

$$\begin{tikzcd}
\check{H}_{n+1}(X,A) & \check{H}_n(A)_{\Gamma(X,A)} \arrow[l, "\delta'^*"'] \arrow[r, "\ell^*"] & \check{H}_n(A) \arrow[ll, bend right, "\delta"'] \\
\check{H}_{n+1}(Y,B) \arrow[u, "f^*"] & \check{H}_n(B)_{\Gamma(Y,B)} \arrow[l, "\delta'^*"] \arrow[r, "\ell^*"'] \arrow[u, "g^*_{\Gamma(X,A)}"'] & \check{H}_n(B) \arrow[u, "g^*"'] \arrow[ll, bend left, "\delta"]
\end{tikzcd}$$

and so $\delta g^* = f^*\delta$.

3.4 Dimension Axiom

Theorem 3.16. *Let P be a one-point space. Then*

$$H_n(P) \cong \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0 \end{cases}, \quad \textit{and} \quad H^n(P) \cong \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0 \end{cases}$$

3.4.1 Proof of Theorem 3.16

Definition 3.7. Let X be a set. If $c : \mathscr{P}(X) \to \mathscr{P}(X)$ is defined by $c(A) = X$ for all nonempty $A \subset X$ and $c(\emptyset) = \emptyset$, we say it is a *trivial closure operator* (or sometimes called *the indiscrete closure operator*).

The only neighborhood for this closure space is $\mathscr{U} = \{X\}$. Thus, $K_{\mathscr{U}}$ is a one-point simplicial and the inverse limit coincides with the homology of the simplex $K_{\mathscr{U}}$.

Let P be a one-point space. Note that the only closure operator is the trivial one, and so the Čech homology

$$\check{H}_*(P) \cong \begin{cases} 0, & n \neq 0 \\ \mathbb{Z}, & n = 0 \end{cases}$$

The same occurs in Čech cohomology.

3.5 Excision Axiom

Theorem 3.17. *Consider a pair (X, A). Let $U \subset X$ be an open set in X, i.e., $i(U) = U$, with $c(U) \subset i(A)$. Let $\iota : (X \setminus U, A \setminus U) \to (X, A)$ be the natural inclusion. Then the induced homomorphisms in (co)homology are isomorphisms, which means that*

$$\iota_* : \check{H}_*(X \setminus U, A \setminus U) \to \check{H}_*(X, A), \quad \textit{and} \quad \iota^* : \check{H}^*(X, A) \to \check{H}^*(X \setminus U, A \setminus U)$$

3.5.1 Proof for Theorem 3.17

For convenience we will write $X' = X \setminus U$ and $A' = A \setminus U$. The condition that U is open implies that $\iota^{-1}(\Gamma(X, A)) \subset \Gamma(X', A')$ is a cofinal subset. Let $(\mathscr{V}', \mathscr{V}'_{A'}) \in \Gamma(X', A')$. Then, by definition of interior cover and using 1.12, we have that

$$X' = \bigcup_{V' \in \mathscr{V}'} i_{X'}(V') = \bigcup_{V' \in \mathscr{V}'} i_X(V' \cup U) \cap X'$$

and

$$A' \subset \bigcup_{V' \in \mathscr{V}'_{A'}} i_{X'}(V') = \bigcup_{V' \in \mathscr{V}'_A} i_X(V' \cup U) \cap X'$$

Now, define $\mathscr{V} := \{V' \cup U | V' \in \mathscr{V}'\}$ and $\mathscr{V}_A := \{V' \cup U | V' \in \mathscr{V}'_{A'}\}$. With this definition, we have that $(\mathscr{V}', \mathscr{V}'_{A'}) = f^{-1}(\mathscr{V}, \mathscr{V}_A)$. Using that U is open, we have that for any $V' \in \mathscr{V}'$ $U = i_X(U) \subset i_X(V' \cup U)$. It follows that

$$X = U \cup (X \setminus U) = U \cup X' = \bigcup_{V' \in \mathscr{V}'} i_X(V' \cup U)$$

and

$$A = U \cup (X \setminus U) = U \cup A' \subset \bigcup_{V' \in \mathscr{V}'_A} i_X(V' \cup U)$$

Therefore, $(\mathscr{V}, \mathscr{V}_A)$ is a interior cover of the pair (X, A).

The condition $c(U) \subset i(A)$ implies that

$$X = i_X(X \setminus U) \cup i_X(A)$$

since $i_X(X \setminus U) = X \setminus c_X(U) \supset X \setminus i_X(A)$.

Define $D \subset \Gamma(X, A)$ as the collection of $(\mathscr{V}, \mathscr{V}_A) \in \Gamma(X, A)$ such that for any $V \in \mathscr{V}$ with $V \cap U \neq \emptyset$, we have that $V \in \mathscr{V}_A$ and $V \subset A$. This subset D is cofinal in $\Gamma(X, A)$, since for any $(\mathscr{W}, \mathscr{W}_A) \in \Gamma(X, A)$ we can define $\mathscr{V}_A := \{V \subset X | V = W \setminus U,\ W \in \mathscr{W}_A, \text{ or } V = W \cap A,\ W \in \mathscr{V}_A\}$ and $\mathscr{V} := \{V \subset X | V = W \setminus U,\ W \in \mathscr{W}, \text{ or } V \in \mathscr{V}_A\}$. Note that for each $W \in \mathscr{W}_A$, we have that

$$\begin{aligned} i_X(W \setminus U) \cup i_X(W \cap A) &= [i_X(W) \cap i_X(X \setminus U)] \cup [i_X(W) \cap i_X(A)] \\ &= i_X(W) \cap [i_X(X \setminus U) \cup i_X(A)] \\ &= i_X(W). \end{aligned}$$

It follows that

$$A \subset \bigcup_{W \in \mathscr{W}_A} i_X(W) = \bigcup_{W \in \mathscr{W}_A} [i_X(W \setminus U) \cup i_X(W \cap A)] = \bigcup_{V \in \mathscr{V}_A} i_X(V)$$

We also have that

$$
\begin{aligned}
X &= i_X(X \setminus U) \cup i_X(A) \\
&= [X \cap i_X(X \setminus U)] \cup [A \cap i_X(A) \cap i_X(A)] \\
&\subset \left[\left(\bigcup_{W \in \mathscr{W}} i_X(W)\right) \cap i_X(X \setminus U)\right] \cup \left[\left(\bigcup_{W \in \mathscr{W}_A} i_X(W)\right) \cap i_X(A)\right] \\
&= \left[\bigcup_{W \in \mathscr{W}} (i_X(W) \cap i_X(X \setminus U))\right] \cup \left[\bigcup_{W \in \mathscr{W}_A} (i_X(W) \cap i_X(A))\right] \\
&= \left[\bigcup_{W \in \mathscr{W}} i_X(W \setminus U)\right] \cup \left[\bigcup_{W \in \mathscr{W}_A} i_X(W \cap A)\right] = \bigcup_{V \in \mathscr{V}} i_X(V)
\end{aligned}
$$

We conclude that in fact $(\mathscr{V}, \mathscr{V}_A)$ is a interior cover of (X, A), and from the definition follows that $(\mathscr{V}, \mathscr{V}_A)$ is a refinement of the given $(\mathscr{W}, \mathscr{W}_A)$.

Let $(\mathscr{V}, \mathscr{V}_A) \in D$. For any $V \in \mathscr{V}$, we have that either $V \subset X \setminus U$ or $V \subset A$. Define $M_{\mathscr{V}}$ as the subcomplex of $K_{\mathscr{V}}$ made of all the simplexes whose vertices are contained in $X \setminus U$. Recall that $\mathscr{V}' := \iota^{-1}(\mathscr{V})$ is a covering for X'. If $V \in \mathscr{V}$ is such that $V \subset X \setminus U$, then we have $V' := \iota^{-1}(V) = V$. Thus, there is a copy of $M_{\mathscr{V}}$ as a subcomplex of $K_{\mathscr{V}'}$ corresponding to the vertices $V' \in \mathscr{V}'$ for which there is $V \in \mathscr{V}$ such that $V' = \iota^{-1}(V) = V$, i.e., $V' \in \mathscr{V}$. We call this copy $M_{\mathscr{V}'}$.

Now, consider the simplicial map $\iota^1_{\mathscr{V}} : (K_{\mathscr{V}'}, L_{\mathscr{V}'_{A'}}) \to (K_{\mathscr{V}}, L_{\mathscr{V}_A})$. Let V' be a vertex in $K_{\mathscr{V}'}$. If V' is a vertex of $M_{\mathscr{V}'}$, then there is a corresponding vertex V in $M_{\mathscr{V}}$ and we can suppose that $\iota^1_{\mathscr{V}}(V') := V$. If V' is not a vertex of $M_{\mathscr{V}'}$, let $V \in \mathscr{V}$ be any vertex such that $V' = \iota^{-1}(V) = V \setminus U$. We have that $V \neq V'$, i.e., $V \cap U \neq \emptyset$, and so, using that $(\mathscr{V}, \mathscr{V}_A) \in D$, we have that $V \in \mathscr{V}_A$ and $V \subset A$. It follows that V' is a vertex of $L_{\mathscr{V}'_{A'}}$ and V is a vertex of $L_{\mathscr{V}_A}$, since $V' \cap A \setminus U = \iota^{-1}(V \cap A) \neq \emptyset$.

Note that the definition of $\iota^1_{\mathscr{V}}$ sends homeomorphically the subcomplex $M_{\mathscr{V}'}$ to $M_{\mathscr{V}}$ and maps $L_{\mathscr{V}'_{A'}}$ into $L_{\mathscr{V}_A}$. Also note that $K_{\mathscr{V}'} = M_{\mathscr{V}'} \cup L_{\mathscr{V}'_{A'}}$ and $K_{\mathscr{V}} = M_{\mathscr{V}} \cup L_{\mathscr{V}_A}$.

The following lemma is equivalent to the excision for the singular (co)homology. It will be used to finish the argument.

Lemma 3.18. *Let K be a simplicial complex, and M, L be subcomplexes whose interiors cover K. Then the inclusion*

$$(M, M \cap L) \overset{f}{\hookrightarrow} (K, L)$$

induces an isomorphism in (co)homology.

Proof. Define $A := K \setminus M$. Then, we have that

$$K \cap L = (K \setminus A) \cap L = L \setminus A$$

and that $c(A) = c(K \setminus M) = K \setminus i(M) \subset i(L)$, since $i(K) \cup i(L) = K$. Using the Excision Axiom for simplicial homology, we have that

$$(M, M \cap L) = (K \setminus A, L \setminus A) \hookrightarrow (K, L)$$

induces isomorphisms on (co)homology. $\square$

Now note that in the following commutative diagram

$$\begin{array}{ccc} H_*(K_{\mathscr{V}'}, L_{\mathscr{V}'_{A'}}) & \xrightarrow{\iota_{\mathscr{V}}} & H_*(K_{\mathscr{V}}, L_{\mathscr{V}_A}) \\ {\scriptstyle f_{\mathscr{V}'}}\uparrow & & \uparrow{\scriptstyle g_{\mathscr{V}}} \\ H_*(M_{\mathscr{V}'}, M_{\mathscr{V}'} \cap L_{\mathscr{V}'_{A'}}) & \xrightarrow[j_{\mathscr{V}}]{} & H_*(M_{\mathscr{V}}, M_{\mathscr{V}} \cap L_{\mathscr{V}_A}) \end{array}$$

the map $j_{\mathscr{V}}$ is an isomorphism since is the induced map of an homeomorphism, and $f_{\mathscr{V}'}, g_{\mathscr{V}}$ are isomorphisms. Thus, $\iota_{\mathscr{V}}$ is an isomorphism.

Similarly we have that there is a corresponding commutative diagram for cohomology

$$\begin{array}{ccc} H^*(K_{\mathscr{V}'}, L_{\mathscr{V}'_{A'}}) & \xleftarrow{\iota_{\mathscr{V}}} & H^*(K_{\mathscr{V}}, L_{\mathscr{V}}) \\ {\scriptstyle f_{\mathscr{V}'}}\downarrow & & \downarrow{\scriptstyle g_{\mathscr{V}}} \\ H^*(M_{\mathscr{V}'}, M_{\mathscr{V}'} \cap L_{\mathscr{V}'_{A'}}) & \xleftarrow[j_{\mathscr{V}}]{} & H^*(M_{\mathscr{V}}, M_{\mathscr{V}} \cap L_{\mathscr{V}_A}) \end{array}$$

Thus, $\iota_{\mathscr{V}}$ is an isomorphism.

Now, define $D' := \iota^{-1}(D) \subset \Gamma(X', A')$. Since $\iota^{-1}(\Gamma(X, A)) \subset \Gamma(X', A')$ is a cofinal subset, we have that D' is a cofinal subset of $\Gamma(X', A')$.

Note that there is an inverse system of isomorphisms $\{\iota_{\mathscr{V}} : H_*(K_{\mathscr{V}'}, L_{\mathscr{V}'_{A'}}) \to H_*(K_{\mathscr{V}}, L_{\mathscr{V}_A})\}$ which induces an isomorphism on the inverse limit. Thus,

$$\check{H}_*(X', A') \xrightarrow{\cong} \varprojlim_{D'}\{H_*(K_{\mathscr{V}'}, L_{\mathscr{V}'_{A'}})\} \xrightarrow{\cong} \varprojlim_{D}\{H_*(K_{\mathscr{V}}, L_{\mathscr{V}_A})\} \xrightarrow{\cong} \check{H}_*(X, A)$$

Similarly, there is a direct system of isomorphisms $\{\iota_{\mathscr{V}} : H^*(K_{\mathscr{V}}, L_{\mathscr{V}_A}) \to H^*(K_{\mathscr{V}'}, L_{\mathscr{V}'_{A'}})\}$ which induces an isomorphism on the direct limit. Thus, in cohomology

$$\check{H}_*(X', A') \xleftarrow{\cong} \varinjlim_{D'}\{H^*(K_{\mathscr{V}'}, L_{\mathscr{V}'_{A'}})\} \xleftarrow{\cong} \varinjlim_{D}\{H^*(K_{\mathscr{V}}, L_{\mathscr{V}_A})\} \xleftarrow{\cong} \check{H}^*(X, A)$$

Chapter 4

Mayer-Vietoris Sequence

In this chapter, we will prove that for any cohomology theory that satisfies the Eilenberg Steenrod axioms there is a Mayer-Vietoris sequence. We only examine the case of cohomology, since the result depends strongly on the exact sequence for a pair, which is not satisfied for Čech homology (even in the topological case). The Mayer-Vietoris sequence is an important tool that allows us to compute the cohomology of a space from the cohomology of two subsets whose interiors cover the space. As mentioned in the introduction, in future work we will generalize these results to obtain the Mayer-Vietoris spectral sequence, and use it for several computations.

We obtain the Mayer-Vietoris Sequence using exact sequences. Thus, we will state and proof this property in cohomology, since the Čech Cohomology satisfies the Exactness axiom and the Čech Homology doesn't. Also, we will prove a general Mayer-Vietoris Sequence, for which we will use triplets (X, A, B), where B is a subspace of A, which also is a subspace of X.

Theorem 4.1. *Given a* cohomology theory (H^*, δ)*, and a triple* (X, A, B) *with inclusions*

$$\iota : (A, B) \to (X, B) \quad \textit{and} \quad j : (X, B) \to (X, A),$$

there is an exact sequence

$$\dots \longrightarrow H^{n-1}(A, B) \xrightarrow{\delta} H^n(X, A) \xrightarrow{j^*} H^n(X, B) \xrightarrow{\iota^*} H^n(A, B) \longrightarrow \dots$$

where δ *is the composite*

$$H^{n-1}(A, B) \to H^{n-1}(A) \to H^n(X, A)$$

Proof. Both maps $\iota : (A, B) \to (X, B)$ and $j : (X, B) \to (X, A)$ induce maps between the exact sequences of the pairs (X, A), (X, B), and (A, B) as seen in the following commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & H^n(X) \arrow[r] \arrow[d] & H^n(A) \arrow[r] \arrow[d] & H^n(X,A) \arrow[r] \arrow[d, "j^*"] & H^{n+1}(X) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^n(X) \arrow[r] \arrow[d] & H^n(B) \arrow[r] \arrow[d] & H^n(X,B) \arrow[r] \arrow[d, "\iota^*"] & H^{n+1}(X) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^n(A) \arrow[r] & H^n(B) \arrow[r] & H^n(A,B) \arrow[r] & H^{n+1}(B) \arrow[r] & \dots
\end{tikzcd}$$

where the rows are exact. Now, we can consider the following commutative diagram by arranging terms on the previous diagram

$$\begin{tikzcd}
 & H^{n-1}(B) \arrow[dr] \arrow[rr, bend left] & & H^n(A,B) \arrow[dr] \arrow[rr, bend left, "\delta"] & & H^{n+1}(X,A) \arrow[dr, "j^*"] \arrow[rr, bend left] & & H^{n+1}(X) \\
H^{n-1}(A) \arrow[ur] \arrow[dr] & & H^n(X,B) \arrow[ur, "\iota^*"] \arrow[dr] & & H^n(A) \arrow[ur] \arrow[dr] & & H^{n+1}(X,B) \arrow[ur] & \\
 & H^n(X,A) \arrow[ur, "j^*"] \arrow[rr, bend right] & & H^n(X) \arrow[ur] \arrow[rr, bend right] & & H^n(B) \arrow[ur] & &
\end{tikzcd}$$

Using the following commutative diagram

$$\begin{tikzcd}
(A,B) \arrow[r, "i"] \arrow[d] & (X.B) \arrow[d, "j"] \\
(A,A) \arrow[r] & (X,A)
\end{tikzcd}$$

and that $H^{[}*](A, A) = 0$, we have that $\iota^* j^* = 0$.

Since the blue, red, and yellow sequences are exact, using the Braid Lemma 5.2 the sequence

$$H^n(X,A) \xrightarrow{j^*} H^n(X,B) \xrightarrow{i^*} H^n(A,B) \xrightarrow{\delta} H^{n+1}(X,A) \xrightarrow{j^*} H^{n+1}(X,A)$$

is exact. $\square$

Theorem 4.2. *Let X_1, X_2 be subspaces of X. The following are equivalent.*

a) *The excision map $(X_1, X_1 \cap X_2) \xrightarrow{k_1} (X_1 \cup X_2, X_2)$ induces an isomorphism of cohomology.*

b) *The excision map $(X_2, X_1 \cap X_2) \xrightarrow{k_2} (X_1 \cup X_2, X_1)$ induces an isomorphism of cohomology.*

c) *The inclusion maps*

$$i_1 : (X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_1 \cap X_2)$$

and

$$i_2 : (X_2, X_1 \cap X_2) \to (X_1 \cup X_2, X_1 \cap X_2)$$

induces epimorphisms on cohomology, and ${i_1}^*, {i_2}^*$ *induce an isomorphism*

$$H^n(X_1 \cup X_2, X_1 \cap X_2) \cong H^n(X_1, X_1 \cap X_2) \oplus H^n(X_2, X_1 \cap X_2)$$

d) *The inclusion maps*

$$j_1 : (X_1 \cup X_2, X_1 \cap X_2) \to (X_1 \cup X_2, X_1)$$

and

$$j_2 : (X_1 \cup X_2, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$$

induces monomorphisms on cohomology, and

$$H^n(X_1 \cup X_2, X_1 \cap X_2) \cong {j_1}^*(H^n(X_1 \cup X_2, X_1)) \oplus {j_2}^*(H^n(X_1 \cup X_2, X_2))$$

e) *For any* $A \subset X_1 \cap X_2$ *there is an exact Mayer-Vietoris sequence*

$$\ldots \to H^n(X_1 \cup X_2, A) \xrightarrow{({g_1}^*, {g_2}^*)} H^n(X_1, A) \oplus H^n(X_2, A) \xrightarrow{{f_1}^* - {f_2}^*} H^n(X_1 \cap X_2, A) \to H^{n+1}(X_1 \cup X_2, A) \to \ldots$$

where $f_\alpha : (X_1 \cap X_2, A) \hookrightarrow (X_\alpha, A)$ *and* $g_\alpha : (X_\alpha, A) \hookrightarrow (X_1 \cup X_2, A)$ *are the natural inclusions.*

f) *For any* $Y \supset X_1 \cup X_2$ *there is an exact Mayer-Vietoris sequence*

$$\ldots \to H^n(Y, X_1 \cup X_2) \xrightarrow{({l_1}^*, {l_2}^*)} H^n(Y, X_1) \oplus H^n(Y, X_2) \xrightarrow{{h_1}^* - {h_2}^*} H^n(Y, X_1 \cap X_2) \to H^{n+1}(Y, X_1 \cup X_2) \to \ldots$$

where $h_\alpha : (Y, X_1 \cap X_2) \hookrightarrow (Y, X_\alpha)$ *and* $l_\alpha : (Y, X_\alpha) \hookrightarrow (Y, X_1 \cup X_2)$ *are the natural inclusions.*

Remark. Similarly as we showed in Lemma 3.18, we have there is a relationship between the Excision Axiom and the pairs $\{X_1, X_2\}$ such that $i\,(X_1) \cup i\,(X_2) = X$ with the inclusion

$$(X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$$

inducing an isomorphism of cohomology.

Proof.

$a) \Rightarrow e)$

Let $A \subset X_1 \cap X_2$. The inclusion the natural inclusion of the triple

$$(X_1, X_1 \cap X_2, A) \hookrightarrow (X_1 \cup X_2, X_2, A)$$

induces the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(X_1 \cup X_2, X_2) \arrow[r] \arrow[d, "{k_1}^*"] & H^n(X_1 \cup X_2, A) \arrow[r] \arrow[d] & H^n(X_2, A) \arrow[r] \arrow[d] & H^{n+1}(X_1 \cup X_2, X_2) \arrow[r] \arrow[d, "{k_1}^*"] & \cdots \\
\cdots \arrow[r] & H^n(X_1, X_1 \cap X_2) \arrow[r] & H^n(X_1, A) \arrow[r] & H^n(X_1 \cap X_2, A) \arrow[r] & H^{n+1}(X_1, X_1 \cap X_2) \arrow[r] & \cdots
\end{tikzcd}$$

Using the lemma 5.3, since ${k_1}^*$ is an isomorphism, there is an induced exact sequence

$$\ldots \to H^n(X_1 \cup X_2, A) \to H^n(X_1, A) \oplus H^n(X_2, A) \to H^n(X_1 \cap X_2, A) \to H^{n+1}(X_1 \cup X_2, A) \to \ldots$$

$e) \Rightarrow c)$

Set $A := X_1 \cap X_2$. Using that $H^n(X_1 \cap X_2, A) = H^n(X_1 \cap X_2, X_1 \cap X_2) = 0$, we have the following exact Mayer-Vietoris sequence

$$\ldots \to 0 \to H^n(X_1 \cup X_2, X_1 \cap X_2) \to H^n(X_1, X_1 \cap X_2) \oplus H^n(X_2, X_1 \cap X_2) \to 0 \to \ldots$$

It follows that

$$H^n(X_1 \cup X_2, X_1 \cap X_2) \xrightarrow[(g_1{}^{[*}]{\cong} , g_2{}^{[}*])] H^n(X_1, X_1 \cap X_2) \oplus H^n(X_2, X_1 \cap X_2)$$

Since $g_\alpha = i_\alpha : (X_\alpha, X_1 \cap X_2) \to (X_1 \cup X_2, X_1 \cap X_2)$, we have that in fact i_1, i_2 induce epimorphisms in cohomology.

$c) \Rightarrow b)$

Consider the following commutative diagram

$$\begin{tikzcd}
(X_1, X_1 \cap X_2) \arrow[dr, "i_1"] \arrow[dd, "k_1"'] & & (X_2, X_1 \cap X_2) \arrow[dl, "i_2"'] \arrow[dd, "k_2"] \\
 & (X_1 \cup X_2, X_1 \cap X_2) \arrow[dl, "j_2"] \arrow[dr, "j_1"'] & \\
(X_1 \cup X_2, X_2) & & (X_1 \cup X_2, X_1)
\end{tikzcd}$$

By hypothesis, ${i_\alpha}^*$ are epimorphism, i.e., $\mathrm{Im}\,({i_\alpha}^*) = H^n(X_\alpha, X_1 \cap X_2)$. Using Theorem 4.1

on the triple $(X_1 \cup X_2, X_\alpha, X_1 \cap X_2)$, there is an exact sequence

$$\cdots \to H^n(X_1 \cup X_2, X_\alpha) \xrightarrow{j_{\alpha *}} H^n(X_1 \cup X_2, X_1 \cap X_2) \xrightarrow{i_{\alpha *}} H^n(X_\alpha, X_1 \cap X_2) \xrightarrow{\delta} H^{n+1}(X_1 \cup X_2, X_\alpha) \to \cdots \tag{4.1}$$

Thus, from the exactness of the previous sequence, we have that $H^n(X_\alpha, X_1 \cap X_2) = \mathrm{Im}\,(i_\alpha{}^*) = \ker(\delta)$, and so $\delta = 0$. Also, from the same sequence, we have that $j_\alpha{}^*$ is an monomorphism, since

$$0 = \mathrm{Im}\,(\delta) = \ker(j_\alpha{}^*)$$

Now, we will show that $k_2{}^*$ is an isomorphism. First, we will prove that $k_2{}^*$ is an epimorphism. Let $a \in H^n(X_2, X_1 \cap X_2)$. Using that $i_2{}^*$ is an epimorphism, there is $b \in H^n(X_1 \cup X_2, X_1 \cup X_2)$ such that $i_2{}^*(b) = a$. Using the hypothesis that

$$H^n(X_1 \cup X_2, X_1 \cap X_2) \cong H^n(X_1, X_1 \cap X_2) \oplus H^n(X_2, X_1 \cap X_2),$$

we have that $i_1{}^*(b) = 0$. Then, $b \in \ker(i_1{}^*) = \mathrm{Im}\,(j_1{}^*)$, and so there is $c \in H^n(X_1 \cup X_2, X_1)$ such that $j_1{}^*(c) = b$. It follows that

$$k_2{}^*(c) = i_2{}^*(j_1{}^*(c)) = i_2{}^*(b) = a$$

Therefore, $k_2{}^*$ is an epimorphism.

Now, we will show that $k_2{}^*$ is a monomorphism. Let $c \in \ker(k_2{}^*)$, i.e.,

$$0 = k_2{}^*(c) = i_2{}^*(j_1{}^*(c)).$$

It follows that $j_1{}^*(c) \in \ker(i_2{}^*)$. Also, using the exact sequence (4.1), we have that $\ker(i_1{}^*) = \mathrm{Im}\,(j_1{}^*)$, and so, using the direct sum assumption, we have that

$$j_1{}^*(c) \in \ker(i_1{}^*) \ker(i_2{}^*) = 0,$$

Since $j_1{}^*$ is a monomorphism, as shown before, we have that $c = 0$, hence $k_2{}^*$ is a monomorphism.

$b) \Rightarrow f)$

Let $Y \supset X_1 \cap X_2$. The inclusion the natural inclusion of the triple

$$(Y, X_2, X_1 \cap X_2) \hookrightarrow (Y, X_1 \cup X_2, X_1)$$

induces the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^n(X_1\cup X_2, X_1) \arrow[r] \arrow[d, "{k_2}^*"] & H^n(Y, X_1\cup X_2) \arrow[r] \arrow[d] & H^n(Y, X_1) \arrow[r] \arrow[d] & H^{n+1}(X_1\cup X_2, X_1) \arrow[r] \arrow[d, "{k_2}^*"] & \cdots \\
\cdots \arrow[r] & H^n(X_2, X_1\cap X_2) \arrow[r] & H^n(Y, X_2) \arrow[r] & H^n(Y, X_1\cap X_2) \arrow[r] & H^{n+1}(X_2, X_1\cap X_2) \arrow[r] & \cdots
\end{tikzcd}$$

Since ${k_2}^*$ are isomorphisms, we can use the lemma 5.3, for which there is an induced exact sequence

$$\cdots \to H^n(Y, X_1\cup X_2) \xrightarrow{({l_1}^*,{l_2}^*)} H^n(Y, X_1)\oplus H^n(Y, X_2) \xrightarrow{{h_1}^*-{h_2}^*} H^n(Y, X_1\cap X_2) \to H^{n+1}(Y, X_1\cap X_2) \to \cdots$$

$f) \Rightarrow d)$

Set $Y := X_1 \cup X_2$. Using that $H^n(Y, X_1\cup X_2) = H^n(X_1\cup X_2, X_1\cup X_2) = 0$, we have the following exact Mayer-Vietoris sequence

$$\ldots \to 0 \to H^n(X_1\cup X_2, X_1) \oplus H^n(X_1\cup X_2, X_2) \to H^n(X_1\cup X_2, X_1\cap X_2) \to 0 \to \ldots$$

It follows that

$$H^n(X_1\cup X_2, X_1) \oplus H^n(X_1\cup X_2, X_2) \xrightarrow[{h_1}^*-{h_2}^*]{\cong} H^n(X_1\cup X_2, X_1\cap X_2)$$

Since $h_\alpha = j_\alpha : (X_1\cup X_2, X_1\cap X_2) \to (X_1\cup X_2, X_\alpha)$, we have that in fact j_1, j_2 induce monomorphisms in cohomology.

$d) \Rightarrow a)$

Consider the following commutative diagram

$$\begin{tikzcd}
(X_1, X_1\cap X_2) \arrow[dr, "i_1"] \arrow[dd, "k_1"'] & & (X_2, X_1\cap X_2) \arrow[dl, "i_2"'] \arrow[dd, "k_2"] \\
& (X_1\cup X_2, X_1\cap X_2) \arrow[dl, "j_2"] \arrow[dr, "j_1"'] & \\
(X_1\cup X_2, X_2) & & (X_1\cup X_2, X_1)
\end{tikzcd}$$

By hypothesis ${j_\alpha}^*$ is an monomorphism, i.e., $0 = \ker({j_\alpha}^*)$. Using Theorem 4.1 on the triple $(X_1\cup X_2, X_\alpha, X_1\cap X_2)$, there is an exact sequence

$$\cdots \to H^n(X_\alpha, X_1\cap X_2) \xrightarrow{\delta} H^n(X_1\cup X_2, X_\alpha) \xrightarrow{j_{\alpha*}} H^n(X_1\cup X_2, X_1\cap X_2) \xrightarrow{i_{\alpha*}} H^n(X_\alpha, X_1\cap X_2) \to \cdots \tag{4.2}$$

we have that $0 = \ker(j_\alpha{}^*) = \mathrm{Im}\,(\delta)$, and so $\delta = 0$. Using again the exact sequence, we have that $i_\alpha{}^*$ is an epimorphism, since

$$H^n(X_\alpha, X_1 \cap X_2) = \ker(\delta) = \mathrm{Im}\,(i_\alpha{}^*)$$

We will show that $k_1{}^*$ is an isomorphism. First, we will show that $k_1{}^*$ is an epimorphism. Let $a \in H^n(X_1, X_1 \cap X_2)$. Using that $i_1{}^*$ is an epimorphism, there is $b \in H^n(X_1 \cup X_2, X_1 \cup X_2)$ such that $i_1{}^*(b) = a$. Using the hypothesis that

$$H^n(X_1 \cup X_2, X_1 \cap X_2) \cong j_1{}^*(H^n(X_1 \cup X_2, X_1)) \oplus j_2{}^*(H^n(X_1 \cup X_2, X_2)),$$

there are $b_1 \in H^n(X_1 \cup X_2, X_1),\ b_2 \in H^n(X_1 \cup X_2, X_2)$ such that

$$b = j_1{}^*(b_1) + j_2{}^*(b_2)$$

It follows that

$$a = i_1{}^*(b) = i_1{}^*(j_1{}^*(b_1) + j_2{}^*(b_2)) = i_2{}^*(j_1{}^*(b_1)) + i_2{}^*(j_2{}^*(b_2)) = i_1{}^*(j_2{}^*(b_1)) = k_1{}^*(b_1),$$

since $i_2{}^* j_2{}^* = 0$. Thus, $k_1{}^*$ is an epimorphism.

Now we will show that $k_1{}^*$ is a monomorphism. Let $c \in \ker(k_1{}^*)$, i.e.,

$$0 = k_1{}^*(c) = i_1{}^*(j_2{}^*(c)),$$

and so,

$$j_2{}^*(c) \in \ker(i_1{}^*) = \mathrm{Im}\,(j_1{}^*),$$

by using the exact sequence (4.2). By the direct sum assumption, we have that

$$\mathrm{Im}\,(j_1{}^*) \cap \mathrm{Im}\,(j_2{}^*) = 0$$

and so $j_2{}^*(c) = 0$. Since $j_2{}^*$ is a monomorphism, we conclude that $c = 0$. Therefore, $k_1{}^*$ is a monomorphism.

$\square$

Definition 4.1. *A triad* $(X; X_1, X_2)$ consists of a space X and two subspaces X_1, X_2 of X. A triad is called *proper* if the inclusions

$$(X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2) \quad \text{and} \quad (X_2, X_1 \cap X_2) \to (X_1 \cup X_2, X_1)$$

induce isomorphisms on cohomology.

If $(X;X_1,X_2)$ and $(Y;Y_1,Y_2)$ are triads, *a continuous function between triads* is a continuous function $f:X\to Y$ such that $f(X_\alpha)\subset Y_\alpha$. We'll denote it by $f:(X;X_1,X_2)\to(Y;Y_1,Y_2)$.

Theorem 4.3. *Let $(X;X_1,X_2)$ and $(Y;Y_1,Y_2)$ be proper triads, and $f:(X;X_1,X_2)\to(Y;Y_1,Y_2)$ be continuous. If $B\subset Y_1\cap Y_2$ and $A\subset X_1\cap X_2$ such that $f(A)\subset B$. Then f induces an homomorphism from the exact Mayer-Vietoris sequence of $\{X_1,X_2;A\}$ into the exact Mayer-Vietoris sequence of $\{Y_1,Y_2;B\}$.*

Similarly, if $V\supset Y_1\cup Y_2$ and $U\supset X_1\cup X_2$ such that $f(U)\subset V$. Then f induces an homomorphism from the exact Mayer-Vietoris sequence of $\{U;X_1,X_2\}$ into the exact Mayer-Vietoris sequence of $\{V;Y_1,Y_2\}$.

Proof. Consider the following commutative diagrams induced by inclusions

$$\begin{array}{ccccc}
H^n(X_1,A) & \longleftarrow & H^n(X_1\cap X_2,A) & \longrightarrow & H^n(X_2,A)\\
\downarrow & & \downarrow & & \downarrow\\
H^n(Y_1,B) & \longleftarrow & H^n(Y_1\cap Y_2,B) & \longrightarrow & H^n(Y_2,B)
\end{array}$$

and

$$\begin{array}{ccccc}
H^n(X_1,A) & \longrightarrow & H^n(X_1\cup X_2,A) & \longleftarrow & H^n(X_2,A)\\
\downarrow & & \downarrow & & \downarrow\\
H^n(Y_1,B) & \longrightarrow & H^n(Y_1\cup Y_2,B) & \longleftarrow & H^n(Y_2,B)
\end{array}$$

These induce the following commutative diagram

$$\begin{array}{ccccc}
H^n(X_1\cap X_2,A) & \longrightarrow & H^n(X_1,A)\oplus H^n(X_2,A) & \longrightarrow & H^n(X_1\cup X_2,A)\\
\downarrow & & \downarrow & & \downarrow\\
H^n(Y_1\cap Y_2,B) & \longrightarrow & H^n(Y_1,B)\oplus H^n(Y_2,B) & \longrightarrow & H^n(Y_1\cup Y_2,B)
\end{array}$$

Now, using that the excision maps

$$(X_1,X_1\cap X_2)\to(X_1\cup X_2,X_2)\quad\text{and}\quad(Y_1,Y_1\cap Y_2)\to(Y_1\cup Y_2,Y_2)$$

induce isomorphisms on cohomology, we have the following commutative diagram

$$\begin{array}{ccccccc}
H^n(X_1\cup X_2,A) & \longrightarrow & H^n(X_1\cup X_2,X_2) & \longrightarrow & H^n(X_1,X_1\cap X_2) & \longrightarrow & H^{n-1}(X_1\cap X_2,A)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
H^n(Y_1\cup Y_2,B) & \longrightarrow & H^n(Y_1\cup Y_2,Y_2) & \longrightarrow & H^n(Y_1,Y_1\cap Y_2) & \longrightarrow & H^{n-1}(Y_1\cap Y_2,B)
\end{array}$$

Therefore,

$$\begin{array}{ccc} H^n(X_1 \cup X_2, A) & \longrightarrow & H^{n-1}(X_1 \cap X_2, A) \\ \downarrow & & \downarrow \\ H^n(Y_1 \cup Y_2, B) & \longrightarrow & H^{n-1}(Y_1 \cap Y_2, B) \end{array}$$

is commutative, and so

$$\begin{array}{ccccccc} H^n(X_1 \cap X_2, A) & \longrightarrow & H^n(X_1, A) \oplus H^n(X_2, A) & \longrightarrow & H^n(X_1 \cup X_2, A) & \longrightarrow & H^{n-1}(X_1 \cap X_2, A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^n(Y_1 \cap Y_2, B) & \longrightarrow & H^n(Y_1, B) \oplus H^n(Y_2, B) & \longrightarrow & H^n(Y_1 \cup Y_2, B) & \longrightarrow & H^{n-1}(Y_1 \cap Y_2, B) \end{array}$$

is also commutative.

The proof is similar for $\{U; X_1, X_2\}, \{V; Y_1, Y_2\}$. $\square$

Chapter 5

Apendix

5.1 Algebra

Proposition 5.1. *Consider the following commutative diagram of groups*

$$\begin{tikzcd}
& B \arrow[dr, "g"] \arrow[dd, bend right, "\phi"'] & \\
A \arrow[ur, "f"] \arrow[dr, "f'"'] & & C \\
& B' \arrow[ur, "g'"'] \arrow[uu, bend right, "\psi"'] &
\end{tikzcd}$$

where ϕ is an isomorphism whose inverse is ψ. If the sequence on top is exact, i.e., $\ker(g) = \mathrm{Im}\,(f)$. *Then the sequence bellow is exact.*

Proof. Since $g'f' = (g\psi)(\phi f) = gf = 0$, we have that $\mathrm{Im}\,(f') \subset \ker(g')$. Now, let $b' \in \ker(g')$. Then, we have that

$$g(\psi(b')) = g'(b') = 0$$

Thus, $\psi(b') \in \ker(g) = \mathrm{Im}\,(f)$, and so there exists $a \in A$ such that $f(a) = \psi(b')$. It follows that

$$f'(a) = \phi(f(a)) = \phi(\psi(b')) = b'$$

Therefore, $\ker(g') \subset \mathrm{Im}\,(f')$. $\square$

Definition 5.1. We say that the following commutative diagram is a braid

$$\begin{tikzcd}
 & A \arrow[rr, bend left, "\alpha"] \arrow[dr, "\epsilon"] & & B \arrow[rr, bend left, "\beta"] \arrow[dr, "\theta"] & & C \arrow[rr, bend left, "\gamma"] \arrow[dr, "\lambda"] & & D \\
E \arrow[ur, "\delta"] \arrow[dr, "\nu"'] & & F \arrow[ur, "\eta"] \arrow[dr, "\rho"] & & G \arrow[ur, "\kappa"] \arrow[dr, "\tau"] & & H \arrow[ur, "\mu"] & \\
 & I \arrow[ur, "\pi"] \arrow[rr, bend right, "\psi"'] & & J \arrow[ur, "\sigma"] \arrow[rr, bend right, "\omega"'] & & K \arrow[ur, "\phi"] & &
\end{tikzcd}$$

Consider the following sequences of the braid

$$E \xrightarrow{\delta} A \xrightarrow{\alpha} B \xrightarrow{\theta} G \xrightarrow{\tau} K \tag{5.1}$$

$$E \xrightarrow{\nu} I \xrightarrow{\psi} J \xrightarrow{\sigma} G \xrightarrow{\kappa} C \xrightarrow{\gamma} D \tag{5.2}$$

$$A \xrightarrow{\epsilon} F \xrightarrow{\rho} J \xrightarrow{\omega} K \xrightarrow{\phi} H \xrightarrow{\mu} D \tag{5.3}$$

$$I \xrightarrow{\pi} F \xrightarrow{\eta} B \xrightarrow{\beta} C \xrightarrow{\lambda} H \tag{5.4}$$

If all four sequences are exact, we say that it is an *exact braid*.

Lemma 5.2 (Braid Lemma)**.** *In order to the braid to be exact, it suffices that the the composite $I \to F \to B$ is zero and that the sequences* (5.1), (5.2), *and* (5.3) *are exact.*

Proof. We'll prove exactness at each step:

1. (Exactness at $I \to F \to B$)
   By hypothesis $\operatorname{Im} \pi \subset \ker \eta$. Let $f \in \ker \eta$. Using the commutativity of the diagram

   $$\sigma(\rho(f)) = \theta(\eta(f)) = \theta(0) = 0,$$

   and so $\rho(f) \in \ker \sigma = \operatorname{Im} \psi$. This means there is $i \in I$ such that $\psi(i) = \rho(f)$. Note that

   $$\rho(f - \pi(i)) = \rho(f) - \rho(\pi(i)) = \rho(f) - \psi(i) = 0,$$

   and so $f - \pi(i) \in \ker \rho = \operatorname{Im} \epsilon$. Thus, there is $a \in A$ such that

   $$\epsilon(a) = f - \pi(i)$$

   Note that $a \in \ker \alpha = \operatorname{Im} \delta$, since

   $$\alpha(a) = \eta(\epsilon(a)) = \eta(f) - \eta(\pi(i)) = 0,$$

because $\eta\pi = 0$. This means there is $e \in E$ such that $\delta(e) = a$. It follows that

$$\pi(\nu(e) + i) = \pi(\nu(e)) + \pi(i) = \epsilon(\delta(e)) + \pi(i) = \epsilon(a) + \pi(i) = f - \pi(i) + \pi(i) = f,$$

and so $f \in \operatorname{Im} \pi$. Therefore $\operatorname{Im} \pi = \ker \eta$.

2. (Exactness at $F \to B \to C$)
First, note that
$$\beta\eta = (\kappa\theta)\eta = \kappa(\theta\eta) = \kappa(\sigma\rho) = (\kappa\sigma)\rho = (0)\rho = 0$$
This means that $\operatorname{Im} \eta \subset \ker \beta$. Now, let $b \in \ker \beta$. Using that
$$\kappa(\theta(b)) = \beta(b) = 0,$$
we have that $\theta(\beta) \in \ker \kappa = \operatorname{Im} \sigma$. Then, there is $j \in J$ such that $\sigma(j) = \theta(b)$. Since
$$\omega(j) = \tau(\sigma(j)) = \tau(\theta(b)) = 0,$$
we have that $j \in \ker \omega = \operatorname{Im} \rho$, and so there is $f \in F$ such that $\rho(f) = j$. It follows that
$$\theta(b - \eta(f)) = \theta(b) - \theta(\eta(f)) = \theta(b) - \sigma(\rho(f)) = \theta(b) - \sigma(j) = 0,$$
which means that $b - \eta(f) \in \ker \theta = \operatorname{Im} \alpha$. Thus, there is $a \in A$ such that $\alpha(a) = b - \eta(f)$. Finally, we have that
$$\eta(\epsilon(a) + f) = \eta(\epsilon(a)) + \eta(f) = \alpha(a) + \eta(f) = b - \eta(f) + \eta(f) = b$$
Therefore $b \in \operatorname{Im} \eta$, and so we conclude that $\ker \beta = \operatorname{Im} \eta$.

3. (Exactness at $B \to C \to H$)
We have that
$$\lambda\beta = \lambda(\kappa\theta) = (\lambda\kappa)\theta = (\phi\tau)\theta = \phi(\tau\theta) = \phi(0) = 0$$
It follows that $\operatorname{Im} \beta \subset \ker \lambda$. Now, let $c \in \ker \lambda \subset C$. Using the commutativity of the diagram, we have that
$$\gamma(c) = \mu(\lambda(c)) = \mu(0) = 0,$$
which means that $c \in \ker \gamma = \operatorname{Im} \kappa$. This means there is $g \in G$ such that $\kappa(g) = c$. Note that $\tau(g) \in \ker \phi = \operatorname{Im} \omega$, since
$$\phi(\tau(g)) = \lambda(\kappa(g)) = \lambda(c) = 0$$

Thus, there is $j \in J$ such that $\omega(j) = \tau(g)$. It follows that

$$\tau(g - \sigma(j)) = \tau(g) - \tau(\sigma(j)) = \tau(g) - \omega(j) = 0,$$

and so $g - \sigma(j) \in \ker \tau$. Then, there is $b \in B$ such that $\theta(b) = g - \sigma(j)$. Using that $\kappa\sigma = 0$, we conclude that

$$\beta(b) = \kappa(\theta(b)) = \kappa(g - \sigma(j)) = \kappa(g) - \kappa(\sigma(j)) = c,$$

and that $c \in \operatorname{Im} \beta$. Therefore, we conclude that $\ker \lambda = \operatorname{Im} \beta$.

□

Lemma 5.3. *Consider the following commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & C''_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f''_{n+1}"] & C'_n \arrow[r, "i_n"] \arrow[d, "f'_n"] & C_n \arrow[r, "p_n"] \arrow[d, "f_n"] & C''_n \arrow[r, "\delta_n"] \arrow[d, "f''_n"] & C''_{n-1} \arrow[r] \arrow[d, "f'_{n-1}"] & \cdots \\
\cdots \arrow[r] & D''_{n+1} \arrow[r, "\partial_{n+1}"] & D'_n \arrow[r, "j_n"] & D_n \arrow[r, "q_n"] & D''_n \arrow[r, "\partial_n"] & D''_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

where the rows are long exact sequences and the vertical maps f''_* are isomorphisms. Then there is an exact sequence

$$\cdots \longrightarrow C'_n \xrightarrow{u_n} C_n \oplus D'_n \xrightarrow{v_n} D_n \xrightarrow{\Delta_n} C'_{n-1} \longrightarrow \cdots$$

where $u_n = (i_n, f'_n),\ v_n = f_n - j_n,\ \Delta_n = \delta_n \phi_n q_n$, and $\phi_n = (f''_n)^{-1}$.

Proof. We will prove the exactness at each step:

1. $(\operatorname{Im} u_n = \ker v_n)$
   Let $c' \in C'_n$. Since the diagram is commutative, we have that $f_n i_n(c') = j_n f'_n(c')$, and so

   $$v_n u_n(c') = v_n(i_n(c'), f'_n(c')) f_n(i_n(c')) - j_n(f'_n(c')) = 0$$

   Then, we have that $v_n u_n = 0$, i.e., $\operatorname{Im} u_n \subset \ker v_n$. Now, let $(c, d') \in \ker v_n \subset C_n \oplus D'_n$. We note that $f_n(c) = j_n(d')$, since $0 = u_n(c, d') = f_n(c) - j_n(d')$. Using that f''_n is an isomorphism, and that

   $$0 = q_n j_n(d') = q_n f_n(c) = f''_n p_n(c) = f''_n(p_n(c)),$$

   we have that $p_n(c) = 0$, i.e., $c \in \ker p_n = \operatorname{Im} i_n$. Consequently, there exists $c' \in C'_n$ such that

$i_n(c') = c$. It follows that

$$j_n(f'_n(c') - d') = j_n(f'_n(c')) - j_n(d') = f_n(i_n(c')) - j_n(d') = f_n(c) - j_n(d') = 0,$$

and using that $\ker j_n = \operatorname{Im} \partial_{n+1}$, there is $d'' \in D''_{n+1}$ such that $\partial_{n+1}(d'') = f'_n(c') - d'$. Finally, we have that

$$\begin{aligned} f'_n(c' - \delta_{n+1}\phi_{n+1}(d'')) &= f'_n(c') - f'_n\delta_{n+1}(\phi_{n+1}(d'')) \\ &= f'_n(c') - \partial_{n+1}f''_{n+1}(\phi_{n+1}(d'')) \\ &= f'_n(c') - \partial_{n+1}(d'') \\ &= f'_n(c') - (f'_n(c') - d') = d' \end{aligned}$$

and that

$$i_n(c' - \delta_{n+1}\phi_{n+1}(d'')) = i_n(c') - i_n\delta_{n+1}(\phi_{n+1}(d'')) = c - 0(\phi_{n+1}(d'')) = c.$$

Thus, we have that $(c, d') = u_n(c' - \delta_{n+1}\phi_{n+1}(d''))$, i.e., $\ker v_n \subset \operatorname{Im} u_n$. Therefore, $\operatorname{Im} u_n = \ker v_n$.

2. $(\operatorname{Im} v_n = \ker \Delta_n)$
Let $(c, d') \in C_n \oplus D'_n$. First, we note that

$$\begin{aligned} \Delta_n v_n(c, d') &= \delta_n\phi_n q_n(f_n(c) - j_n(d')) \\ &= \delta_n\phi_n q_n f_n(c) - \delta_n\phi_n q_n(j_n(d')) \\ &= \delta_n\phi_n f''_n p_n(c) - \delta_n\phi_n(0(d')) \\ &= \delta_n p_n(c) = 0, \end{aligned}$$

which means that $\Delta_n v_n = 0$, i.e., $\operatorname{Im} v_n \subset \ker \Delta_n$. Now, let $d \in \ker \Delta_n \subset D_n$. Since $0 = \Delta_n(d) = \delta_n(\phi_n q_n(d))$, we have that $\phi_n q_n(d) \in \ker \delta_n = \operatorname{Im} p_n$, for which there is $c \in C_n$ such that $p_n(c) = \phi_n q_n(d)$. It follows that

$$q_n(f_n(c) - d) = f''_n p_n(c) - q_n(d) = f''_n\phi_n q_n(d) - q_n(d) = q_n(d) - q_n(d) = 0.$$

Thus, we have that $f_n(c) - d \in \ker q_n = \operatorname{Im} j_n$, and so there exists $d' \in D'_n$ such that $j_n(d') = f_n(c) - d$. It follows that $d \in \operatorname{Im} v_n$, since

$$d = f_n(c) - j_n(d') = v_n(c, d').$$

Therefore, we have that $\ker \Delta_n \subset \operatorname{Im} v_n$, and with this we conclude that $\operatorname{Im} v_n = \ker \Delta_n$.

3. $(\operatorname{Im}\Delta_n = \ker u_{n-1})$
   Let $d \in D_n$. Then, we have that

$$\begin{aligned} u_{n-1}\Delta_n(d) &= (i_{n-1}(\delta_n\phi_n q_n(d)), f'_{n-1}(\delta_n\phi_n q_n(d))) \\ &= (0(\phi_n q_n(d)), \partial_{n-1} f''_n \phi_n q_n(d)) \\ &= (0, \partial_{n-1} q_n(d)) = (0,0), \end{aligned}$$

   and so $u_{n-1}\Delta_n = 0$, i.e., $\operatorname{Im}\Delta_n \subset \ker u_{n-1}$. Now, let $c' \in \ker u_{n-1} \subset C'_{n-1}$, i.e., we have that $(0,0) = u_{n-1}(c') = (i_{n-1}(c'), f'_{n-1}(c'))$. Since $i_{n-1}(c') = 0$ and that $\ker i_{n-1} = \operatorname{Im}\delta_{n-1}$, there exists $c'' \in C''_n$ such that $\delta_n(c'') = c'$. We also note that

$$\partial_{n-1} f''_n(c'') = f'_{n-1}\delta_n(c'') = f_{n-1}(c') = 0,$$

   and, using that $\ker \partial_{n-1} = \operatorname{Im} q_n$, there exists $d \in D_n$ such that $q_n(d) = f''_n(c'')$. Therefore, we have that

$$\Delta_n(d) = \delta_n\phi_n q_n(d) = \delta_n\phi_n f''_n(c'') = \delta_n(c'') = c'.$$

   We conclude that $\ker u_{n-1} \subset \operatorname{Im}\Delta_n$, and so $\operatorname{Im}\Delta_n = \ker u_{n-1}$.

$\square$

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