CIMAT Centro de Investigación en Matemáticas, A.C.

MORITA EQUIVALENCE ON THE GENERATORS OF LOCALIZATION AS A MODULE OVER DIFFERENTIAL OPERATORS

T E S I S

Que para obtener el grado de

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Presenta

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Introduction

Given a field, there is a correspondence between geometric objects in K^n and algebraic objects in $R = K[x_1, \ldots, x_n]$. Specifically, given an ideal I in R there is a corresponding variety $\mathrm{V}_I = \{x \in K^n \mid f(x) = 0, f \in I\}$. On the other hand, given a variety $V \subseteq K^n$ there is a corresponding ideal $\mathrm{I}_V = \{f \in R \mid f(x) = 0, x \in V\}$. The relation between these is given by

$$\mathrm{V}_{\mathrm{I}_V} = V, \quad \text{and} \quad I \subseteq \mathrm{I}_{\mathrm{V}_I}.$$

If the field is algebraically closed, the Hilbert Nullstellensatz Theorem establishes that $V(I) = V(\sqrt{I})$.

A variety can be abstracted from sets in K^n to subsets in $\operatorname{Spec}(R)$. In this context, $V(I) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$. The Zariski topology in $\operatorname{Spec}(R)$ is defined by its closed sets; $\{V(I)\}_{I \subseteq R}$. If I is principal, generated by $f \in R$, we say that the variety $V(I)$ is a hypersurface. We set

$$U_f = \operatorname{Spec}(R) \setminus V(f) \cong \operatorname{Spec}(R_f),$$

which is an open set in this topology. The collection of open sets $\{U_f\}_{f \in R}$ form a basis for the Zariski topology. In fact, using that $V(f+g) = V(f) \cap V(g)$, we deduce that

$$\operatorname{Spec}(R) \setminus V(I) = \bigcup_{f \in I} U_f.$$

Using the above, we can study the hypersurface $V(f)$ by investigating the properties of R_f. This presents a challenge because, in general, R_f is not a finitely generated $R-$module. In this work, we study the action of higher order differential operators on R_f as a mean to put additional structure to R_f.

In characteristic zero the differential operators on the polynomial ring are described by the Weyl Algebra

$$R\langle \partial_1, \ldots, \partial_n \rangle \subseteq \operatorname{Hom}_K(R, R),$$

where $\partial_i = \dfrac{\partial}{\partial x_i}$. In contrast, in prime characteristic this is not enough for describe it. If $R = \mathbb{F}_p[x_1, \ldots, x_n]$, then $\partial_i^p(x_i^p) = p! = 0$ in $\mathbb{F}_p$. We can define the following differential operators for $\alpha \in \mathbb{N}^n$

$$\frac{1}{\alpha!}\partial_{\underline{x}}^{\alpha}(\underline{x}^{\beta}) = \begin{cases} \binom{\beta}{\alpha}\underline{x}^{\beta-\alpha}, & \alpha_i \leq \beta_i \ \forall i \\ 0, & \text{otherwise}, \end{cases}$$

where $\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}$, and $\underline{x}^{\beta} = x_1^{\beta_1} \cdots x_n^{\beta_n}$. Thus, we have that

$$D_{R|\mathbb{F}_p} = R\left\langle \frac{1}{\alpha!}\partial_{\underline{x}}^{\alpha} \;\middle|\; \alpha \in \mathbb{N}^n \right\rangle.$$

In characteristic zero, we have that R_f is a finitely generated $D_{R|K}$-module. Furthermore, R_f is cyclic by the existence of Bernstein-Sato polynomial [**1, 2, 3**]. This is a nonzero polynomial $b_f(s) \in \mathbb{Q}[s]$ such that there is $\delta(s) \in D_{R|K}[s]$ with

$$\delta(t)f^{t+1} = b_f(t)f^t, \text{ for every } t \in \mathbb{Z}.$$

The main goal of this thesis is to describe the work of Àlvarez Montaner, Blickle, and Lyubeznik [**4**]. The following Theorem is a stronger statement in prime characteristic, which does not use the Bernstein-Sato polynomial.

THEOREM 0.1 (Theorem 3.13 (see also [**4**])). *Let R be a regular finitely generated commutative algebra over an $F-$finite regular local ring A of prime characteristic $p > 0$. Let $f \in R$ be a nonzero element. Then, $R_f = D_R \cdot \frac{1}{f}$.*

This approach uses characterization of differential operators in prime characteristic, Frobenius decent, Morita equivalence, and $p-$th roots of ideals. One of the consequences of the , we obtain the corresponding result for direct summands of rings [**5**].

In chapter 1, we provide background on differential operators in general. We approach the topic in a categorical view. At the end of the first chapter we give some properties of differential operators over rings of prime characteristic.

On the chapter 2, our goal is to develop the Morita Equivalence and give the background for it. In this chapter we mainly use the book [**6**] as reference. From this, there are two examples that we are interested in: the Morita Equivalence between a ring and the $n \times n-$matrix with entries on the ring; and the Frobenius Descent. The former is used to define the latter along with the Corollary 1.33 and Proposition 2.41.

Finally, in chapter 3, we prove the main theorem in this work. In order to obtein this result, we explore $p-$th roots of ideals. We relate this idelas to the action of $D_{R|A}$ on R_f. At the end of the chapter, we use this theorem to obtain the corresponding result for direct summands.

CHAPTER 1

Differential Operators

In this chapter we go through basic definitions and properties of differential operators. Our focus is to understand them as functors and when they have a useful and simple presentation. Another focus is in the case of prime characteristic and how we can express them in a more computable way. In this chapter we consider a ring to be commutative with unit, unless stated otherwise.

1. Definition and first properties

In general, in this section we will write $A \subseteq R$ to say that R is an $A-$algebra. If we refer to a module without specifying if it's either left or right, then the result is equally valid for both right and left and the actual side of the action does not play part in the proof.

In this section we define differential operators

We begin this section by recalling some useful isomorphism for

LEMMA 1.1. *Let A and R be any ring. If M is an $A-$module, Q is an $R-$module, and N is both an $A-$ module and an $R-$module. Then the homomorphism*

$$\eta : \mathrm{Hom}_R\left(N \otimes_A M, Q\right) \to \mathrm{Hom}_A\left(M, \mathrm{Hom}_R\left(N, Q\right)\right)$$

defined by

$$\left[[\eta(f)](m)\right](n) = f(n \otimes m),$$

is an isomorphism whose inverse λ is given by

$$\left[\lambda(g)\right](n \otimes m) = \left[g(m)\right](n).$$

PROOF. We proceed to show that the compositions $\eta\lambda$ and $\lambda\eta$ are the respective identities. Let $f \in \mathrm{Hom}_R\left(N \otimes_A M, Q\right)$ and $g \in \mathrm{Hom}_A\left(\mathrm{Hom}_R\left(N, Q\right)\right)$. Then

$$[\lambda\eta(f)](n \otimes m) = [\lambda[\eta(f)]](n \otimes m) = [[\eta(f)](m)](n) = f(n \otimes m),$$

and

$$[[\eta\lambda(g)](m)](n) = [[\eta[\lambda(g)]](m)](n) = [\lambda(g)](n \otimes m) = [g(m)](n),$$

which shows that the statement is true. □

COROLLARY 1.2. *Let R be an $A-$algebra. Then, for any $R-$modules M and N, there are isomorphisms*

$$\mathrm{Hom}_R\left(R \otimes_A R, M\right) \cong \mathrm{Hom}_A\left(R, \mathrm{Hom}_R\left(R, M\right)\right) \cong \mathrm{Hom}_A\left(R, M\right).$$

PROOF. The first isomorphism is given by Lemma 1.1 by the map

$$[[\eta(\varphi)](m)](r) = \varphi(r \otimes m)$$

and the second isomorphism is given by $\eta_\varphi(m) \mapsto [\eta_\varphi(m)](1) = \varphi(1 \otimes m)$. □

Note that $\operatorname{Hom}_R(R \otimes_A M, N)$ is canonically an $R \otimes_A R-$module by the action

$$(a \otimes b \cdot \varphi)(r \otimes m) = \varphi[(a \otimes b)(r \otimes m)] = \varphi(ar \otimes bs).$$

On the other hand, there are two possible actions of R on $\operatorname{Hom}_A(M, N)$:

i) $(a \cdot \delta)(m) = a\delta(m)$.

ii) $(b \cdot \delta)(m) = \delta(bm)$.

By default we are considering the first action, but we can encode both of them as an $R \otimes_A R$ action by

$$(a \otimes b) \cdot \delta(m) = a\delta(bm).$$

If we consider the homomorphism $R \to R \otimes_A R$ defined by $r \mapsto r \otimes 1$, then the restriction of R scalars coincides with the $R \otimes_A R$ action.

Now, we have that the isomorphism in Corollary 1.2 is actually an isomorphism of $R \otimes_A R-$modules. Let $\delta \in \operatorname{Hom}_A(M, N)$ and $\varphi \in \operatorname{Hom}_R(R \otimes_A M, N)$ be such that $\delta(m) = \varphi(1 \otimes m)$. Then, we have that

$$[a \otimes b \cdot \delta](m) = [a \otimes b \cdot \varphi](1 \otimes m) = \varphi(a \otimes bm) = [[\eta(\varphi)](bm)](a) = a[[\eta(\varphi)](bm)](1) = a\delta(bm)$$

since $[\eta(\varphi)](bm)$ is an $R-$linear homomorphism.

Definition 1.3. Given a pair of rings $A \subseteq R$, and $R-$module M. We say that $\partial \in \operatorname{Hom}_A(R, M)$ is an $A-$*linear derivation* if

$$\partial(rs) = r\partial(s) + s\partial(r), \text{ for all } r, s \in R.$$

We denote the set of all $A-$linear derivations from R to M by $\operatorname{Der}_A(M)$.

Definition 1.4. From now on, we write $P_{R|A} = R \otimes_A R$. Consider $\mu : P_{R|A} \to R$ defined by $\mu(r \otimes s) = rs$. If $\Delta_{R|A} = \ker \mu$, then the module of $A-$*linear Kähler differentials on* R is

$$\Omega_{R|A} = \frac{\Delta_{R|A}}{\Delta^2_{R|A}}.$$

Furthermore, there is a natural map $d : R \to \Omega_{R|A}$, called the *universal differential*, defined by

$$d(r) = (1 \otimes r - r \otimes 1) + \Delta^2_{R|A} \in \Omega_{R|A}.$$

Proposition 1.5. *Let* $\Delta_{R|A}$ *as before. The set*

$$\{1 \otimes r - r \otimes 1 \mid r \in R\}$$

generates $\Delta_{R|A}$ *as an* $R-$*module.*

Proof. Let $r \in R$. Then,

$$\mu(1 \otimes r - r \otimes 1) = \mu(1 \otimes r) - \mu(r \otimes 1) = 1r - r1 = 0.$$

Thus, we have that $\langle 1 \otimes r - r \otimes 1 | r \in R\rangle \subseteq \Delta_{R|A}$.

Now, let $w = \sum_{i=1}^{n} r_i \otimes s_i \in \Delta_{R|A}$. Recall that $0 = \mu(w) = \sum_{i=1}^{n} r_i s_i$. Thus,

$$w = w + 0 \otimes 1 = \sum_{i=1}^{n} r_i \otimes s_i + \left(\sum_{i=1}^{n} r_i s_i\right) \otimes 1 = \sum_{i=1}^{n} (r_i \otimes s_i - r_i s_i \otimes 1) = \sum_{i=1}^{n} r_i (1 \otimes s_i - s_i \otimes 1),$$

and so, $w \in \langle 1 \otimes r - r \otimes 1 | r \in R\rangle$. $\square$

Definition 1.6. Define the adjoin homomorphism of $A-$modules by

$$\begin{aligned} \mathrm{ad}: R &\longrightarrow P_{R|A} \\ r &\mapsto 1\otimes r - r\otimes 1. \end{aligned}$$

We have that the homomorphism $\phi : P_{R|A} \to \Delta_{R|A}$ of $R-$modules given by $\phi(r\otimes s) = r\,\mathrm{ad}(s)$ is a splitting of the inclusion $\Delta_{R|A} \hookrightarrow P_{R|A}$. Furthermore, we have that

$$\begin{aligned} \mathrm{ad}(rs) &= 1\otimes rs - rs\otimes 1 \\ &= 1\otimes rs - r\otimes s + r\otimes s - rs\otimes 1 \\ &= (1\otimes r - r\otimes 1)(1\otimes s) + (1\otimes s - s\otimes 1)(r\otimes 1) \\ &= \mathrm{ad}(r)(1\otimes s) + \mathrm{ad}(s)(r\otimes 1) \end{aligned}$$

Proposition 1.7. *For a pair of rings $A \subseteq R$ and a $R-$module M, there is an isomorphism of $R-$modules*

$$\mathrm{Der}_A(M) \cong \mathrm{Hom}_R\left(\Omega_{R|A}, M\right)$$

which is functorial on M.

Proof. Let $\delta \in \mathrm{Der}_A(M) \subseteq \mathrm{Hom}_A(R, M)$. As seen in Corollary 1.2, there is a unique map $\varphi \in \mathrm{Hom}_A\left(P_{R|A}, M\right)$ such that $\varphi(r\otimes s) = r\delta(s)$. It follows that

$$\varphi\,\mathrm{ad}(r) = \varphi(1\otimes r - r\otimes 1) = 1\delta(r) - r\delta(1) = d(r),$$

and that

$$\begin{aligned} \varphi(\mathrm{ad}(r)\,\mathrm{ad}(s)) &= \varphi((1\otimes r - r\otimes 1)(1\otimes s - s\otimes 1)) \\ &= \varphi(1\otimes rs - s\otimes r - r\otimes s + rs\otimes 1) \\ &= 1\delta(rs) - s\delta(r) - r\delta(s) + rs\delta(1) \\ &= \delta(rs) - s\delta(r) - r\delta(s) \\ &= 0 \end{aligned}$$

since δ is a derivation. Therefore, we conclude that there is a unique map $\phi : \Delta_{R|A}/\Delta^2_{R|A} \to M$ that satisfies

$$\phi d(r) = \phi(1\otimes r - r\otimes 1 + \Delta^2_{R|A}) = \varphi(1\otimes r - r\otimes 1) = \delta(r).$$

□

Definition 1.8. Let M be any $R-$module. For any $r \in R$, we use μ_r to denote the multiplication by r on the module, i.e., $\mu_r(m) = rm$.

Definition 1.9. Let R be a an $A-$algebra. We define the *$A-$linear differential operators on* R of order ar most i, denoted by $D^i_{R|A}$, inductively by

- $D^0_{R|A} = \mathrm{Hom}_R(R, R)$.
- $D^i_{R|A} = \left\{\delta \in \mathrm{Hom}_A(R, R) \;\middle|\; [\delta, \mu_r] \in D^{i-1}_{R|A},\ \text{for all } r \in R\right\}$,

where $[\delta, \mu_r] = \delta\mu_r - \mu_r\delta$.

The *collection of all the $A-$linear differential operators on* R is

$$D_{R|A} = \bigcup_{i\in\mathbb{N}} D^i_{R|A}.$$

Proposition 1.10. *$D_{R|A}$ is both an $R-$module and a $P_{R|A}-$module with the actions defined by*

$$r \cdot \delta = \mu_r \delta, \quad \textit{and} \quad r \otimes s \cdot \delta = \mu_r \delta \mu_s.$$

Proof. First, we note that for any $\delta \in D^0_{R|A}$ there is $t \in R$ such that $\delta = \mu_t$. Thus,

$$r \cdot \delta = \mu_r \delta = \mu_r \mu_t = \mu_{rt} \in D^0_{R|A},$$

and

$$r \otimes s \cdot \delta = \mu_r \delta \mu_s = \mu_r \mu_t \mu_s = \mu_{rts} \in D^0_{R|A}.$$

Now, suppose there is some $n \in \mathbb{N}$ such that for all $\delta \in D^n_{R|A}$, and $r \in R$, we have that $r \cdot \delta \in D^n_{R|A}$. Let $\delta \in D^{n+1}_{R|A}$ and $r, s \in R$. Then, we have that

$$\begin{aligned}
[(r \cdot \delta), \mu_s] &= (\mu_r \delta)\mu_s - \mu_s(\mu_r \delta) \\
&= \mu_r \delta \mu_s - \mu_r \mu_s \delta \\
&= \mu_r [\delta, \mu_s] \in D^n_{R|A}
\end{aligned}$$

since $[\delta, \mu_s] \in D^n_{R|A}$.

Similarly, suppose there is some $n \in \mathbb{N}$ such that for all $\delta \in D^n_{R|A}$, and $r \otimes s \in P_{R|A}$, we have that $r \otimes s \cdot \delta \in D^n_{R|A}$. Let $\delta \in D^{n+1}_{R|A}$, $r \otimes s \in P_{R|A}$, and $t \in R$. Then,

$$\begin{aligned}
[[(r \otimes s) \cdot \delta], \mu_t] &= [(r \otimes s) \cdot \delta]\mu_t - \mu_t[(r \otimes s) \cdot \delta] \\
&= [\mu_r \delta \mu_s]\mu_t - \mu_t[\mu_r \delta \mu_s] \\
&= \mu_r \delta \mu_t \mu_s - \mu_r \mu_t \delta \mu_s \\
&= \mu_r [\delta \mu_t - \mu_t \delta]\mu_s \\
&= (r \otimes_A s) \cdot [\delta, \mu_t] \in D^n_{R|A}
\end{aligned}$$

since $[\delta, \mu_t] \in D^n_{R|A}$.

We showed that every $D^i_{R|A}$ is an $R-$module and $P_{R|A}-$module, and so $D_{R|A}$ is too. $\square$

Proposition 1.11. *If $\alpha \in D^m_{R|A}$ and $\beta \in D^n_{R|A}$, then $\alpha\beta \in D^{m+n}_{R|A}$.*

Proof. We proceed by induction on $k = m + n$. Let $r \in R$. The case $k = 0$ and $k = 1$ follow from the action $r \otimes s \cdot \alpha = \mu_r \alpha \mu_s \in D^{0+1+0}_{R|A}$, because $\mu_r, \mu_s \in D^0_{R|A}$.

Now, suppose the conclusion is valid for some $k \in \mathbb{N}$. Let $m, n \in \mathbb{N}$ such that $m + n = k + 1$, and $\alpha \in D^m_{R|A}$ and $\beta \in D^n_{R|A}$. Then,

$$\begin{aligned}
[\alpha\beta, \mu_r] &= (\alpha\beta)\mu_r - \mu_r(\alpha\beta) \\
&= \alpha\beta\mu_r - \alpha\mu_r\beta + \alpha\mu_r\beta - \mu_r\alpha\beta \\
&= \alpha(\beta\mu_r - \mu_r\beta) + (\alpha\mu_r - \mu_r\alpha)\beta \\
&= \alpha[\beta, \mu_r] + [\alpha, \mu_r]\beta \in D^{m+n-1}_{R|A},
\end{aligned}$$

because $\alpha[\beta, \mu_r] \in D^{m+(n-1)}_{R|A}$ and $[\alpha, \mu_r]\beta \in D^{(m-1)+n}_{R|A}$. $\square$

As a consequence of the previous proposition

Proposition 1.12. *If R is an $A-$algebra, then*

$$D^1_{R|A} \cong R \oplus \mathrm{Der}_A(R).$$

PROOF. For any $\delta \in D^1_{R|A}$, we take $\delta' = \delta - \mu_{\delta(1)} \in D^1_{R|A}$. Note that

$$\delta'(1) = \delta(1) - \mu_{\delta(1)}(1) = \delta(1) - \delta(1) = 0$$

It follows, for any $r \in R$, that

$$[\delta', \mu_r](1) = \delta'(r) - r\delta'(1) = \delta'(r)$$

Since $[\delta', \mu_r] \in D^0_{R|A}$, we conclude that $[\delta', \mu_r] = \mu_{\delta'(r)}$. Now, for any $r, s \in R$, we have that

$$\begin{aligned}\delta'(rs) &= \delta'\mu_r(s)\\ &= [\delta'\mu_r - \mu_r\delta' + \mu_r\delta'](s)\\ &= [\mu_{\delta'(r)} + \mu_r\delta'](s)\\ &= \delta'(r)s + r\delta'(s),\end{aligned}$$

and so, $\delta' \in \mathrm{Der}_A(R)$. Therefore, $D^1_{R|A} = \mathrm{Hom}_R(R, R) \oplus \mathrm{Der}_A(R) \cong R \oplus \mathrm{Der}_A(R)$. □

DEFINITION 1.13. Let $A \subseteq R$ be a pair of rings, and both M and N be $R-$module. We define the $A-$*linear differential operators from* M *to* N, inductively by

- $D^0_{R|A}(M, N) = \mathrm{Hom}_R(M, N)$.
- $D^i_{R|A}(M, N) = \left\{\delta \in \mathrm{Hom}_A(M, N) \;\middle|\; [\delta, \mu_r] = \mathrm{ad}(r)\delta \in D^{i-1}_{R|A}(M, N),\ \text{for all } r \in R\right\}$.

and so

$$D_{R|A}(M, N) = \bigcup_{i \in \mathbb{N}} D^i_{R|A}(M, N).$$

For simplicity, we write $D^i_{R|A}(N) = D^i_{R|A}(R, N)$, and $D_{R|A}(N) = D_{R|A}(R, N)$. Note that we also can write $D^0_{R|A}(M) = \{\delta \in \mathrm{Hom}_A(R, M) \,|\, \mathrm{ad}(r)\delta = 0,\ \text{for all } r \in R\}$.

$$D^0_{R|A}(M) = \{\delta \in \mathrm{Hom}_A(R, M) \,|\, \mathrm{ad}(r)\delta = 0,\ \text{for all } r \in R\}$$

In general $D_{R|A}(M, N)$ is no longer a noncommutative ring, unless $M = N$. Nevertheless, the composition of differential operators is still defined.

PROPOSITION 1.14. *Let* M, N, *and* L *be* $R-$*modules. If* $\delta \in D^i_{R|A}(M, N)$ *and* $\partial \in D^k_{R|A}(N, L)$, *then* $\partial\delta \in D^{i+k}_{R|A}(M, L)$.

PROOF. The proof is analogous to the proof of Proposition 1.11. □

One of our goals in this chapter is to understand both $D_{R|A}(M, N)$ as a module and as a functor. We need to show that $D_{R|A}$ satisfies functorial properties that are natural on both M and N.

PROPOSITION 1.15. *Let* $\phi : M \to M'$ *and* $\varphi : N \to N'$ *be homomorphisms between* $R-$*modules. Let* $\delta \in D^i_{R|A}(M, N)$, *and* $\partial = \varphi\delta\phi$, *as shown in the commutative diagram*

$$\begin{array}{ccc} M' & \xrightarrow{\partial} & N' \\ {\scriptstyle\phi}\downarrow & & \uparrow{\scriptstyle\varphi} \\ M & \xrightarrow{\delta} & N \end{array}$$

Then $\delta \in D(M', N')$.

PROOF. Recall that $\phi \in \operatorname{Hom}_R(M, M') = D^0_{R|A}(M, M')$, and $\varphi \in \operatorname{Hom}_R(N, N') = D^0_{R|A}(N, N')$. Now, using Proposition 1.14 we have that

$$\partial = \varphi\delta\phi \in D^{0+i+0}_{R|A}(M', N').$$

□

We showed there are functorial properties between modules over the same ring. Furthermore, if we fixed the modules, then there are functorial properties for different rings.

PROPOSITION 1.16. *Let* $A \xrightarrow{g} S \xrightarrow{f} R$ *be ring homomorphisms. If* M *and* N *are* R *modules, then for every* $i \in \mathbb{N}$

$$D^i_{R|S}(M, N) \subseteq D^i_{R|A}(M, N) \subseteq D^i_{S|A}(M, N).$$

PROOF. First note that

$$\operatorname{Hom}_R(M, N) \subseteq \operatorname{Hom}_S(M, N) \subseteq \operatorname{Hom}_A(M, N).$$

Thus, we have that

$$D^0_{R|S}(M, N) = \operatorname{Hom}_R(M, N) = D^0_{R|A}(M, N),$$

and that

$$D^0_{R|A}(M, N) = \operatorname{Hom}_R(M, N) \subseteq \operatorname{Hom}_S(M, N) = D^0_{S|A}(M, N).$$

We proceed by induction. Suppose that the result is true for some $n \in \mathbb{N}$. Let $\delta \in D^{n+1}_{R|S}(M, N)$. Using the definition, we have that

$$\delta \in \operatorname{Hom}_S(M, N) \subseteq \operatorname{Hom}_A(M, N),$$

such that $\operatorname{ad}(r)\delta \in D^n_{R|S}(M, N) \subseteq D^n_{R|A}(M, N)$, by induction hypothesis. It follows that $\delta \in D^{n+1}_{R|A}(M, N)$.

Now, let $\delta \in D^{n+1}_{R|A}(M, N)$. Since $\delta \in \operatorname{Hom}_A(M, N)$, we need to check that $\operatorname{ad}(s)\delta \in D^n_{S|A}(M, N)$, for every $s \in S$. Using that

$$\operatorname{ad}(s)\delta(m) = \delta(s \cdot m) - s \cdot \delta(m) = \delta(f(s)m) - f(s)\delta(m) = \operatorname{ad}(f(s))\delta(m),$$

we have by induction hypothesis

$$\operatorname{ad}(s)\delta = \operatorname{ad}(f(s))\delta \in D^n_{R|A}(M, N) \subseteq D^n_{S|A}(M, N).$$

Therefore $\delta \in D^{n+1}_{S|A}(M, N)$, and so we have that

$$D^{n+1}_{R|S}(M, N) \subseteq D^{n+1}_{R|A}(M, N) \subseteq D^{n+1}_{S|A}(M, N).$$

□

2. Principal parts

In this section we develop tools to compute differential operators as

LEMMA 1.17. *Given a ring* S*, an ideal* $J \subseteq S$ *and an* $S-$*module* M*, we have that*

$$S/J \otimes_S M \cong \frac{M}{JM}.$$

Proof. Consider the exact sequence

$$J \to S \to S/J \to 0.$$

Using that $-\otimes_S M$ is a right exact functor, we have the following exact sequence

$$J \otimes_S M \to S \otimes_S M \to S/J \otimes_S M \to 0.$$

Recall that for any $s \in S$ and $m \in M$, we have that

$$s \otimes m = 1 \otimes sm.$$

It follows that $S \otimes_S M \cong M$, and that $J \otimes M \cong JM$. Thus, by universal property of the cokernel, we conclude that

$$S/J \otimes_S M \cong \frac{M}{JM}.$$

□

Definition 1.18. Take $\Delta_{R|A}$ as in definition 1.4. We define the *module of $i-$principal parts of R over A* as

$$P^i_{R|A} = \frac{P_{R|A}}{\Delta^{i+1}_{R|A}}.$$

Now, given an $R-$module M, set

$$P_{R|A}(M) = R \otimes_A M, \quad \text{and} \quad P^i_{R|A}(M) = P_{R|A}(M) \otimes_{P_{R|A}} P^i_{R|A}.$$

Using the lemma 1.17, we have that

$$P^i_{R|A}(M) \cong \frac{R \otimes_A M}{\Delta^{i+1}_{R|A}(R \otimes_A M)}.$$

Define the map $d_M : M \to P_{R|A}(M)$ by $d_M(m) = 1 \otimes_A m$. Let $\rho^i_M : P_{R|A}(M) \to P^i_{R|A}(M)$ the natural projection. Then, we take $d^i_M = \rho^i_M d_M$. Note that both $P_{R|A}(M)$ and $P^i_{R|A}(M)$ are $P_{R|A}-$modules.

Proposition 1.19. *Let R be an $A-$algebra, and let M and N be $R-$modules. There is an isomorphism*

$$\begin{aligned} \operatorname{Hom}_R\left(P^i_{R|A}(M), N\right) &\to D^i_{R|A}(M, N) \\ \phi \quad &\mapsto \quad \phi d^i_M. \end{aligned}$$

Proof. Recall, from Corollary 1.2, that $\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_R\left(P_{R|A}(M), N\right)$ is a $P_{R|A}-$module. Note that $\delta \in D^i_{R|A}(M, N)$ if and only if, for any $r \in R$,

$$\operatorname{ad}(r)\delta \in D^{i-1}_{R|A}(M, N).$$

By continuing this process, we have that $\delta \in D^i_{R|A}(M, N) \subseteq \operatorname{Hom}_A(R, M)$ if and only if

$$\operatorname{ad}(r_0) \cdots \operatorname{ad}(r_i)\delta = 0, \text{for any } r_0, \ldots, r_i \in R.$$

Using Corollary 1.2, we can take $\varphi \in \operatorname{Hom}_R\left(P_{R|A}(M), N\right)$ such that $\delta(m) = \varphi(1 \otimes m)$. Thus,

$$0 = [\operatorname{ad}(r_0) \cdots \operatorname{ad}(r_i)\delta](m) = [\operatorname{ad}(r_0) \cdots \operatorname{ad}(r_i)\varphi](1 \otimes m) = \varphi(\operatorname{ad}(r_0) \cdots \operatorname{ad}(r_i)(1 \otimes m)).$$

It follows that $\varphi(\Delta^{i+1}_{R|A}(R \otimes_A M)) = 0$, and so, there exists $\phi : P^i_{R|A}(M) \to N$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "d_M"] \arrow[dr, "\delta"'] & P_{R|A}(M) \arrow[r, "\rho^i_M"] \arrow[d, dashed, "\varphi"] & P^i_{R|A}(M) \arrow[dl, "\phi"] \\ & N & \end{tikzcd}.$$

Since all the correspondences are unique, and that $\rho^i_M d_M = d^i_M$, we showed the desired isomorphism. $\square$

Corollary 1.20. *With the same assumptions as before,*

$$D_{R|A}(M, N) = \varinjlim_{i \in \mathbb{N}} \operatorname{Hom}_R\left(P^i_{R|A}(M), N\right)$$

Proposition 1.21. *We have that*

$$D^1_{R|A}(M) \cong M \oplus \operatorname{Der}_A(M).$$

Proof. Consider the exact sequence

$$0 \to \Delta_{R|A}/\Delta^2_{R|A} \to P_{R|A}/\Delta^2_{R|A} \to \frac{P_{R|A}/\Delta^2_{R|A}}{\Delta_{R|A}/\Delta^2_{R|A}} \cong \frac{P_{R|A}}{\Delta_{R|A}} \to 0.$$

We define

$$\begin{aligned} \varphi : P^1_{R|A} = P_{R|A}/\Delta^2_{R|A} &\to \Delta_{R|A}/\Delta^2_{R|A} = \Omega_{R|A} \\ r \otimes s + \Delta^2_{R|A} &\mapsto -s(1 \otimes r - r \otimes 1) + \Delta^2_{R|A}. \end{aligned}$$

Note that

$$\begin{aligned} \varphi(1 \otimes r - r \otimes 1 + \Delta^2_{R|A}) &= \varphi(1 \otimes r + \Delta^2_{R|A}) - \varphi(r \otimes 1 + \Delta^2_{R|A}) \\ &= -r(1 \otimes 1 - 1 \otimes 1) - (-1(1 \otimes r - r \otimes 1)) + \Delta^2_{R|A} \\ &= 1 \otimes r - r \otimes 1 + \Delta^2_{R|A}, \end{aligned}$$

and so φ is a split map.

Recall that $\mu : P_{R|A} \to R$ is surjective. Thus,

$$P^0_{R|A} = P_{R|A}/\Delta_{R|A} \cong R.$$

Now, we have that

$$\begin{aligned} D^1_{R|A}(M) &\cong \operatorname{Hom}_R\left(P^1, M\right) \\ &\cong \operatorname{Hom}_R\left(R \oplus \Omega_{R|A}, M\right) \\ &\cong \operatorname{Hom}_R(R, M) \oplus \operatorname{Hom}_R\left(\Omega_{R|A}, M\right) \\ &\cong M \oplus \operatorname{Der}_{R|A}(M). \end{aligned}$$

$\square$

3. Prime characteristic in differential operators

We

We recall properties of the Frobenius map and look at the effect it has on differential operators

PROPOSITION 1.22. *If R is an $A-$algebra, then the functor $D^i_{R|A}(-) = D^i_{R|A}(R, -)$ is a left exact functor.*

PROOF. As we showed in Proposition 1.19, we have that $D^i_{R|A}(-) = D^i_{R|A}(R, -) \cong \operatorname{Hom}_R\left(P^i_{R|A}, -\right)$, which is a left exact functor. □

PROPOSITION 1.23. *Let R be an $A-$algebra, and $W \subseteq R$ a multiplicative subset. Then*
$$W^{-1}P^i_{R|A} \cong P^i_{W^{-1}R|A} \cong P^i_{W^{-1}R|(W\cap A)^{-1}A}.$$

PROOF. First, we have
$$W^{-1}P^i_{R|A} = (W \otimes 1)^{-1}\left(\frac{R \otimes_A R}{\Delta^{i+1}_{R|A}}\right).$$
Using that
$$1 \otimes w = w \otimes 1 + (1 \otimes w - w \otimes 1) \in w \otimes 1 + \Delta_{R|A},$$
and that
$$P^i_{W^{-1}R|A} = \frac{W^{-1}R \otimes_A W^{-1}R}{\Delta^{i+1}_{W^{-1}R|A}} \cong (W \otimes 1)^{-1}(1 \otimes W)^{-1}\left(\frac{R \otimes_A R}{\Delta^{i+1}_{R|A}}\right)$$
we conclude that in fact $W^{-1}P^i_{R|A} \cong P^i_{W^{-1}R|A}$. For the second equality, note that
$$\frac{a}{w}\frac{r_1}{w_1} \otimes \frac{r_2}{w_2} = \frac{ar_1}{ww_1} \otimes \frac{wr_2}{ww_2} = \frac{wr_1}{ww_1} \otimes \frac{ar_2}{ww_2} = \frac{r_1}{w_1} \otimes \frac{a}{w}\frac{r_2}{w_2}.$$
It follows that, there is an $A-$isomorphism
$$\begin{aligned} W^{-1}R \otimes_A W^{-1}R &\to W^{-1}R \otimes_{(W\cap A)^{-1}A} W^{-1}R \\ \frac{r_1}{w_1} \otimes \frac{r_2}{w_2} &\mapsto \frac{r_1}{w_1} \otimes \frac{r_2}{w_2}. \end{aligned}$$
□

THEOREM 1.24. *Let R be an $A-$algebra. If $W \subseteq R$ is a multiplicative closed subset, and M is an $R-$module. Then*
$$W^{-1}D_{R|A}(R, M) \cong D_{W^{-1}R|A}(W^{-1}R, W^{-1}M) \cong D_{W^{-1}R|(W\cap A)^{-1}A}(W^{-1}R, W^{-1}M)$$

PROOF. From the Proposition 1.23, we have that
$$\begin{aligned} D_{W^{-1}R|A}(W^{-1}R, W^{-1}M) &\cong \operatorname{Hom}_{W^{-1}R}\left(P^i_{W^{-1}R|A}, W^{-1}M\right) \\ &\cong \operatorname{Hom}_{W^{-1}R}\left(P^i_{W^{-1}R|(W\cap A)^{-1}A}, W^{-1}M\right) \\ &\cong \operatorname{Hom}_{W^{-1}R}\left(W^{-1}P^i_{R|A}, W^{-1}M\right). \end{aligned}$$
From Proposition, we have that
$$\operatorname{Hom}_{W^{-1}R}\left(P^i_{W^{-1}R|(W\cap A)^{-1}A}, W^{-1}M\right) \cong D^i_{W^{-1}R|(W\cap A)^{-1}A}(R, M).$$

Note that $P^i_{R|A}$ is finitely generated and, using that R is Noetherian, is also finitely presented. It follows from Lemma 2.36 that

$$\operatorname{Hom}_{W^{-1}R}\left(W^{-1}P^i_{R|A}, W^{-1}M\right) \cong W^{-1}\operatorname{Hom}_R\left(P^i_{R|A}, M\right) \cong W^{-1}D_{R|A}(R, M).$$

$\square$

In the following we follow the results of [**7**], with some differences in the proofs.

Definition 1.25. Let A be a ring of characteristic $p > 0$. Then the map

$$\begin{aligned} F_A : A &\to A \\ a &\mapsto a^p \end{aligned}$$

is an homomorphism of rings, and we call it the Frobenius map. Now, for any $e \geq 0$ define ${}^eA$ as $A-$algebra given by the Frobenius map $F^e_A : A \to {}^eA$. Note that the Frobenius map $F_A : {}^eA \to {}^{e+1}A$ is a homomorphism of $A-$algebras because, for any $\lambda \in A$ and $a \in A$,

$$F_A(\lambda \cdot a) = F_A(\lambda^{p^e} a) = \lambda^{p^{e+1}} a^p = \lambda \cdot a^p = \lambda \cdot F_A(a).$$

Definition 1.26. Let A be a ring of characteristic p, and let R be an $A-$algebra. If $F_A : A \to A$ is the Frobenius map, we consider A to be an $A-$algebra with the structure provided by this map. We define

$$R^{(p|A)} = {}^1A \otimes_A R$$

where A acts as follows $a \cdot 1 \otimes r = F_A(a) \otimes r = 1 \otimes ar$. We can make $R^{(p|A)}$ an $A-$algebra by $a \mapsto a \otimes 1$. With this structure, the map

$$\begin{aligned} F_{R|A} : R^{(p|A)} &\to R \\ a \otimes r &\mapsto ar^p \end{aligned}$$

is an $A-$algebra homomorphism. Thus, we have the commutative diagram of $A-$algebras

$$\begin{tikzcd} R \arrow[r, "G_{R|A}"] & R^{(p|A)} \arrow[r, "F_{R|A}"] & R \\ A \arrow[u] \arrow[r, "F_A"] & A \arrow[u] \arrow[ur] & \end{tikzcd}$$

where $G_{R|A}(r) = 1 \otimes r$, for $r \in R$. Recursively, we define

$$R^{(p^{e+1}|A)} = \left(R^{(p^e|A)}\right)^{p|A} \cong {}^eA \otimes_{{}^{e-1}A} R^{p^{e-1}|A} \cong {}^eA \otimes_A R$$

and write the iteration of the relative Frobenius by $F^e_{R|A} : R^{(p^e|A)} \to R$.

Lemma 1.27. *Let A be a ring of characteristic $p > 0$, and let R be an $A-$algebra. If M and N are $R-$modules, then*

$$D^{p^e-1}_{R|A}(M, N) \subseteq \operatorname{Hom}_{R^{(p^e|A)}}(M, N).$$

Proof. First, set $S_n = R^{(p^e|A)}$. Since S_n is an $A-$algebra via the map $\lambda \mapsto \lambda \otimes 1$, k we have that S_n is generated as an $A-$module by $\{1 \otimes r \mid r \in R\}$. As seen in Proposition 1.5, we have that $\Delta_{S_n|A}$ is generated as an S_n-module by $\{1 \otimes s - s \otimes 1 \mid s \in S\}$. Combining both of these observations, we have that

$$\{1 \otimes (1 \otimes r) - (1 \otimes r) \otimes 1 \mid r \in R\}$$

generates $\Delta_{S_n|A}$ as an $A-$module. If $G_n = F^e_{R|A} \otimes_A F^e_{R|A}$. Then,

$$G_n(1 \otimes (1 \otimes r) - (1 \otimes r) \otimes 1) = 1 \otimes r^{p^e} - r^{p^e} \otimes 1 = (1 \otimes r^{p^e} - r^{p^e} \otimes 1)^{p^e}.$$

Thus, $G_n(\Delta_{S_n|A}) \subseteq \Delta^{p^e}_{R|A}$. Now, let $\delta \in D^{p^e-1}_{R|A}(M, N)$. Using Theorem 1.2, we have that

$$\mathrm{Hom}_{S_n}\left(P_{S_n|A}(M), N\right) \cong \mathrm{Hom}_A\left(M, N\right) \cong \mathrm{Hom}_R\left(P_{R|A}(M), N\right),$$

and so there are $\psi \in \mathrm{Hom}_{S_n}\left(P_{S_n|A}(M), N\right)$ and $\varphi \in \mathrm{Hom}_R\left(P_{R|A}(M), N\right)$ such that $\psi(1 \otimes m) = \delta$

$$\psi(1 \otimes m) = \delta(m) = \varphi(1 \otimes m).$$

Since M and N are S_n-modules by restriction of scalars, we have that

$$\psi(s \otimes m) = s \cdot \psi(1 \otimes m) = F^e_{R|A}(s)\delta(m) = F^e_{R|A}(s)\varphi(1 \otimes m) = \varphi(F^e_{R|A}(s) \otimes m)$$

It follows that for any $s_a, s_b \in S_n$ and $m \in M$

$$\begin{aligned}
(s_a \otimes s_b)\delta(m) &= \psi((s_a \otimes s_b)(1 \otimes m)) \\
&= \psi(s_a \otimes F^e_{R|A}(s_b)m) \\
&= \varphi(F^e_{R|A}(s_a) \otimes F^e_{R|A}(s_b)m) \\
&= \varphi((F^e_{R|A}(s_a) \otimes F^e_{R|A}(s_b))(1 \otimes m)) \\
&= \varphi(G_n(s_a \otimes s_b)(1 \otimes m)).
\end{aligned}$$

As seen in the Proposition 1.19, we have that $\varphi(\Delta^{p^e}_{R|A}P_{R|A}M) = 0.$ and so

$$\Delta_{S_n|A}\delta(m) = \varphi(G_n(\Delta_{S_n|A})(1 \otimes m)) \subseteq \varphi(\Delta^{p^e}_{R|A}(1 \otimes m)) = 0.$$

Therefore, we conclude that

$$(s \otimes 1 - 1 \otimes s)\delta = 0,$$

i.e., $s\delta(m) = \delta(sm)$, for any $s \in S_n$. □

PROPOSITION 1.28. *Suppose that R is generated by $r_1, \ldots, r_\ell$ as an $R^{(p|A)}$-algebra. Then, for every $e \in \mathbb{N}$, R is finitely generated by the same elements as $R^{(p^e|A)}$-algebra.*

PROOF. We proceed by induction. Suppose that for some $e \in \mathbb{N}$, R is finitely generated by the same elements as $R^{(p^e|A)}$-algebra. Recall that $R^{(p^e|A)} \cong {}^eA \otimes_A R$. Using the hypothesis for a given $r \in R$ there are $a_\alpha \in A$ $r_\alpha \in R$ such that

$$r = \sum_\alpha (a_\alpha \otimes r_\alpha) \cdot r_1^{\alpha_1} \cdots r_\ell^{\alpha_\ell} = \sum_\alpha a_\alpha r_\alpha^p r_1^{\alpha_1} \cdots r_\ell^{\alpha_\ell}$$

By inductive hypothesis, there are $a_{\alpha\beta} \in A$ and $r_{\alpha\beta} \in R$ such that

$$r_\alpha = \sum_{\beta \in B} (a_{\alpha\beta} \otimes r_{\alpha\beta}) \cdot r_1^{\beta_1} \cdots r_\ell^{\beta_\ell} = \sum_{\beta \in B} a_{\alpha\beta} r_{\alpha\beta}^{p^e} r_1^{\beta_1} \cdots r_\ell^{\beta_\ell}.$$

It follows that

$$\begin{aligned} r &= \sum_{\alpha \in A} a_\alpha r_\alpha^p r_1^{\alpha_1} \cdots r_\ell^{\alpha_\ell} \\ &= \sum_{\alpha \in A} a_\alpha \left(\sum_{\beta \in B} a_{\alpha\beta} r_{\alpha\beta}^{p^e} r_1^{\beta_1} \cdots r_\ell^{\beta_\ell} \right)^p r_1^{\alpha_1} \cdots r_\ell^{\alpha_\ell} \\ &= \sum_{\alpha \in A} \sum_{\beta \in B} a_\alpha a_{\alpha\beta}^p r_{\alpha\beta}^{p^{e+1}} r_1^{\alpha_1 + p\beta_1} \cdots r_\ell^{\alpha_\ell + p\beta_\ell}. \end{aligned}$$

Therefore, R is generated as an $R^{(p^{e+1}|A)}$ algebra by $r_1, \ldots, r_\ell$. □

LEMMA 1.29. *Suppose that R is generated by ℓ elements as an $R^{(p|A)}$-algebra. Then*

$$\operatorname{Hom}_{R^{(p^e|A)}}(M, N) \subseteq D_{R|A}^{\ell p^e - 1}, \text{ for each } e \in \mathbb{N}.$$

PROOF. Fix $e \in \mathbb{N}$. Let $S_e = R^{(p^e|A)}$. Consider $\Delta_{R|S_e}$. Using the identity

$$\operatorname{ad}(rs) = 1 \otimes rs - rs \otimes 1 = (1 \otimes r - r \otimes 1)(1 \otimes s) + (1 \otimes s - s \otimes 1)(r \otimes 1),$$

we have that

$$\operatorname{ad}\left(\sum_\alpha s_\alpha r_1^{\alpha_1} \ldots r_\ell^{\alpha_\ell} \right) = \sum_\alpha s_\alpha \operatorname{ad}(r_1^{\alpha_1} \ldots r_\ell^{\alpha_\ell}) \in (\operatorname{ad}(r_1), \ldots \operatorname{ad}(r_\ell)).$$

Since each $\operatorname{ad}(r_i) \in \Delta_{R|S_e}$, we conclude that

$$\Delta_{R|S_e} = (\operatorname{ad}(r_1), \ldots, \operatorname{ad}(r_\ell)).$$

as an ideal of $P_{R|S_e}$. Note that

$$(1 \otimes r_i - r_i \otimes 1)^{p^e} = 1 \otimes r_i^{p^e} - r_i^{p^e} \otimes 1 = 1 \otimes r_i^{p^e} - (1 \otimes r_i) 1 \otimes 1 = 1 \otimes r_i^{p^e} - 1 \otimes (1 \otimes r_i) 1 = 0,$$

and so, by pigeonhole principle, $\Delta_{R|S_e}^{\ell p^e} = 0$. Thus, $P_{R|S_e}^{\ell p^e - 1} = R \otimes_{S_e} R$. It follows that

$$\begin{aligned} \operatorname{Hom}_{S_e}(M, N) &\cong \operatorname{Hom}_R\left(P_{R|S_e}(M), N \right) \\ &\cong \operatorname{Hom}_R\left(P_{R|S_e}^{\ell p^e - 1}(M), N \right) \\ &\cong D_{R|S_e}^{\ell p^e - 1}(M, N). \end{aligned}$$

Using Proposition 1.16 and the commutative diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & R \\ & S_e \arrow[ur] & \end{tikzcd}$$

we have that

$$D_{R|S_e}^{\ell p^e - 1}(M, N) \subseteq D_{R|A}^{\ell p^e - 1}(M, N).$$

□

THEOREM 1.30. *Let R be an $A-$algebra, with A a ring of characteristic $p > 0$. If the relative Frobenius $F_{R|A} : R^{(p|A)} \to A$ is finite, then*

$$D_{R|A}(M, N) = \bigcup_{e \in \mathbb{N}} \operatorname{Hom}_{R^{(p^e|A)}}(M, N).$$

PROOF. From lemma 1.27, we have that

$$D_{R|A}(M,N)=\bigcup_{e\in\mathbb{N}} D^{p^e-1}_{R|A}(M,N)\subseteq \bigcup_{e\in\mathbb{N}} \mathrm{Hom}_{R^{(p^e|A)}}(M,N).$$

Now, using that the relative Frobenius $F_{R|A}:R^{(p|A)}\to A$ is finite, we have that R is finitely generated by ℓ elements as an $R^{(p|A)}-$algebra, for some $\ell\in\mathbb{N}$. Thus, by lemma 1.29, we conclude that

$$\bigcup_{e\in\mathbb{N}} \mathrm{Hom}_{R^{(p^e|A)}}(M,N)\subseteq \bigcup_{e\in\mathbb{N}} D^{\ell p^e-1}_{R|A}(M,N)=D_{R|A}(M,N).$$

□

DEFINITION 1.31. Let R be a ring of prime characteristic $p>0$. If $F:R\to R$ is the Frobenius map, then, for each $e\in\mathbb{N}$, define

$$D^{(e)}_R=\mathrm{End}_{R^{p^e}}(R),$$

where $R^{p^e}=F^e_R(R)\subseteq R$.

PROPOSITION 1.32. *Let $\mathbb{k}$ be a perfect field of prime characteristic $p>0$. If R is a $K-$algebra, then $R\overset{W_{R|K}}{\cong} R^{p|K}$.*

PROOF. First, note that $W_{R|K}$ is injective. We need to show that $W_{R|K}$ is surjective. Using that K is perfect, we have $K^p=F_K(K)=K$. It follows for any $k\otimes r\in R^{p|K}$ there is $k'\in K$ such that $k=F_K(k)$, and so

$$k\otimes r=F_K(k')\otimes r=1\otimes k'r=W_{R|K}(k'r).$$

It follows that the composite

$$R\xrightarrow{W_{R|K}} R^{p|K}\xrightarrow{F_{R|A}} R$$

is the Frobenius map $F_R:R\to R$. □

COROLLARY 1.33. *If R is essentially of finite type over $\mathbb{k}$, a perfect field of prime characteristic $p>0$. Then,*

$$D_{R|\mathbb{k}}=\bigcup_{e\in\mathbb{N}} \mathrm{Hom}_{R^{p^e}}(R,R).$$

PROOF. This result is a consequence of Theorem 1.30, and Proposition 1.32. Let $S=\mathbb{k}[x_1,\dots,x_n]$ such that

$$R\cong W^{-1}(\mathbb{k}[x_1,\dots,x_n]/I),$$

for some ideal $I\subseteq S$ and a multiplicative set $W\subseteq S/I$. Since $\mathbb{k}$ is perfect, we have that

$$F_{S|\mathbb{k}}(S^{p|\mathbb{k}})=F_S(S)=S^p=K^p[x_1^p,\dots,x_n^p]=K[x_1^p,\dots,x_n^p].$$

It follows that the set $\{x_1^{a_1}\cdots x_n^{a_n}\mid 0\le a_i<p, i=1,\dots,n\}$ is a generating set of S as an S^p module. Thus, the map $F_{S|\mathbb{k}}$ is finite. Consider the commutative diagram

$$\begin{array}{ccccc} S & \longrightarrow & S^{p|\mathbb{k}} & \xrightarrow{F_{S|\mathbb{k}}} & S \\ \downarrow & & \downarrow & & \downarrow \\ R & \longrightarrow & R^{p|\mathbb{k}} & \xrightarrow{F_{R|\mathbb{k}}} & R \end{array}.$$

Since the vertical maps are induced by a projection and a localization, we conclude that $F_{R|\Bbbk}$ is also finite and, by Theorem 1.30,

$$D_{R|\Bbbk} = \bigcup_{e\in\mathbb{N}} \operatorname{Hom}_{R^{(p^e|\Bbbk)}}(R, R) = \bigcup_{e\in\mathbb{N}} \operatorname{Hom}_{R^{p^e}}(R, R) = \bigcup_{e\in\mathbb{N}} D^{(e)}_R.$$

$\square$

CHAPTER 2

Morita equivlance

In this chapter, our goal is to define the Frobenius Descent by using Morita Equivalences. We first explain what is a Morita Equivalence, for which we start with a background in Category Theory. Then, we develop the necessary tools to prove Morita's Theorem. After that, we show the Morita Equivalence between a ring R and $\mathrm{Mat}_n(R)$ the $n \times n-$matrices with entries on R. Finally, we define the Frobenius Descent with the use of the later Morita equivalence.

1. Categorical Prelimenaries

Definition 2.1. We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that

$$FG \overset{\eta}{\simeq} \mathbb{1}_{\mathcal{D}}, \quad \text{and} \quad GF \overset{\epsilon}{\simeq} \mathbb{1}_{\mathcal{C}}.$$

Equivalently, for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$ there are natural isomorphisms

$$\eta_C : GF(C) \to C \quad \text{and} \quad \epsilon_D : FG(D) \to D.$$

Definition 2.2. Let $F : \mathcal{C} \to \mathcal{D}$, be a functor:

(1) We say that F is *full*, if for any objects $C_1, C_2 \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2))$$

is surjective.

(2) We say that F is *faithful*, if for any $C_1, C_2 \in \mathcal{C}$, the map

$$\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2))$$

is injective.

(3) We say that F is *essentially surjective*, if for any $D \in \mathcal{D}$ there is $C \in \mathcal{C}$ such that $F(C) \cong D$.

Theorem 2.3. *If a functor $F : \mathcal{C} \to \mathcal{D}$ is a category equivalence, then F is fully faithful and essentially surjective.*

Proof. We show the statement by parts.

(1) [Essentially surjective.] For any $D \in \mathcal{D}$, there is $C = G(D)$ such that $F(C) = F(G(D)) = FG(D) \cong D$.

(2) [Faithful.] Since $GF(C_i) \cong C_i$, we have that the composite

$$\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2)) \to \mathrm{Hom}_{\mathcal{C}}(GF(C_1), GF(C_2))$$

is an isomorphism. Thus, we have that the first map is injective.

(3) [Full.] For convenience, we write $H = GF$. Note that for any $C \in \mathcal{C}$ we have the commutative diagram

$$\begin{tikzcd} H(C) \arrow[r, "\eta_C"] & C \\ H(H(C)) \arrow[u, "\eta_{H(C)}"] \arrow[r, "H(\eta_C)"'] & H(C) \arrow[u, "\eta_C"'] \end{tikzcd} .$$

Thus, we have that

$$\eta_C \eta_{H(C)} = \eta_C H(\eta_C).$$

Since η_C is an isomorphism, we have that $\eta_{H(C)} = H(\eta_C)$.

Let $g \in \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2))$. Define $f = \eta_{C_2} G(g) \eta_{C_1}^{-1}$ as shown in

$$\begin{tikzcd} H(C_1) \arrow[r, "G(g)"] \arrow[d, "\eta_{C_1}"'] & H(C_2) \arrow[d, "\eta_{C_2}"] \\ C_1 \arrow[r, dotted, "f"'] & C_2 \end{tikzcd} .$$

Note that the map $G(g)$ induces the following commutative diagram

$$\begin{tikzcd} H(H(C_1)) \arrow[r, "H(G(g))"] \arrow[d, "\eta_{H(C_1)}"'] & H(H(C_2)) \arrow[d, "\eta_{H(C_2)}"] \\ H(C_1) \arrow[r, "G(g)"'] & H(C_2) \end{tikzcd} .$$

Thus, we have that

$$\begin{aligned} H(f) &= H(\eta_{C_2}) H(G(g)) H(\eta_{C_1}^{-1}) \\ &= \eta_{H(C_2)} H(G(g)) \eta_{H(C_1)}^{-1} \\ &= G(g) \end{aligned}$$

Using that G is faithful, we conclude that $F(f) = g$, because G is a category equivalence. So we conclude that F is full.

□

Definition 2.4. Let R be a ring. A left $R-$module is an Abelian group $(M, +)$ along with a left action of R into M such that for all $r, s \in R$ and $m, n \in M$, we have:

(1) $(r + s) \cdot (m + n) = r \cdot m + r \cdot n + s \cdot m + s \cdot n$.
(2) $(rs) \cdot m = r \cdot (s \cdot m)$.
(3) $1_R \cdot m = m$.

Similarly, a right $R-$module is an Abelian group $(M, +)$ along with a right action of R into M

(1) $(m + n) \cdot (r + s) = r \cdot m + r \cdot n + s \cdot m + s \cdot n$.
(2) $m \cdot (sr) = (m \cdot s) \cdot r$.
(3) $m \cdot 1_R = m$.

We'll add the notation of $R-\mathrm{Mod}$ to the category of left R modules, and $\mathrm{Mod}-R$ to the category of right R modules.

Definition 2.5. Let R and S be two rings. Then an $(R-S)-$bimodule is an Abelian group $(M,+)$ such that

(1) M is both a left $R-$module and a right $S-$module.
(2) For every $r \in R$, $s \in S$, and $m \in M$ we have that

$$(rm)s = r(ms).$$

Definition 2.6. Let $f, g : M \to N$ be two homomorphisms between $R-$modules. We define:

- The *equalizer* of f and g is an $R-$module E and a homomorphism $e : E \to M$, which satisfy $fe = ge$, such that for any homomorphism $\alpha : A \to M$, if $f\alpha = g\alpha$, then there exists a unique homomorphism $u : A \to E$ such that $eu = \alpha$.

$$\begin{tikzcd} E \arrow[r, "e"] & M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N \\ & A \arrow[u, "\alpha"'] \arrow[ul, dotted, "u"] & \end{tikzcd}$$

- The *coequalizer* of f and g is an $R-$module Q and a homomorphism $q : N \to Q$, which satisfy $qf = qg$, such that for any homomorphism $\beta : N \to B$, if $\beta f = \beta g$, then there exists a unique homomorphism $v : Q \to B$ such that $vq = \beta$.

$$\begin{tikzcd} M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N \arrow[r, "q"] \arrow[d, "\beta"'] & Q \arrow[dl, dotted, "v"] \\ & B & \end{tikzcd}$$

Proposition 2.7. *The* equalizer *of f and g is given by*

$$\mathrm{Eq}(f, g) = \mathrm{Ker}(f - g) = \{m \in M \mid f(m) = g(m)\},$$

along with the inclusion $e : \mathrm{Eq}(f, g) \hookrightarrow M$.

Proof. Directly from the definition, we have that $(f - g)e = 0$, and so $f\iota = g\iota$. Now, if $h : A \to M$ is such that $fh = gh$, then $(f - g)h = 0$. By the universal property of the kernel, there is a homomorphism $u : A \to \mathrm{Eq}(f, g)$, such that $eu = h$. □

Proposition 2.8. *The* coequalizer *of f and g is*

$$\mathrm{Coeq}(f, g) = \mathrm{Coker}(f - g)$$

along with the projection $q : N \to \mathrm{Coeq}(f, g)$.

Proof. Using that $q(f - g) = 0$, we have that $qf = qg$. Let $h : N \to A$ be such that $hf = hg$. It follows that $(f - g)h = 0$. By the universal property of the cokernel, there is a homomorphism $v : \mathrm{Coeq}(f, g) \to A$, such that $vq = h$. □

Proposition 2.9. *Let $f, g : M \to N$ be two homomorphisms between $R-$modules. If $F : R - \mathrm{Mod} \to S - \mathrm{Mod}$ is an equivalence of categories, then*

$$F\left(\mathrm{Eq}(f, g)\right) \cong \mathrm{Eq}(F(f), F(g)) \quad \textit{and} \quad F\left(\mathrm{Coeq}(f, g)\right) \cong \mathrm{Coeq}(F(f), F(g)).$$

Proof. Let $\alpha : A \to F(M)$ and $\beta : F(N) \to B$ such that $F(f)\alpha = F(g)\alpha$ and $\beta F(f) = \beta F(g)$. By Theorem 2.3 F is essentially surjective, thus there are $R-$modules A' and B' such that $F(A') \overset{\phi}{\cong} A$ and $B \overset{\psi}{\cong} F(B')$. Let

$$\alpha' = \alpha\phi, \quad \text{and} \quad \beta' = \psi\beta$$

Using that F is full, by Theorem 2.3, there are $\alpha'' : A' \to M$ and $\beta'' : N \to B'$ such that $F(\alpha'') = \alpha'$ and $F(\beta'') = \beta'$. Using the universal properties of the equalizer and co-equalizer there are $u' : A' \to \mathrm{Eq}(f,g)$ and $v' : \mathrm{Coeq}(f,g) \to B'$ such that $\alpha'' = eu'$, and $\beta'' = v'q$.

$$\begin{tikzcd}
F\left(\mathrm{Eq}(f,g)\right) \arrow[r, "F(e)"] & F\left(M\right) \arrow[r, shift left, "F(f)"] \arrow[r, shift right, "F(g)"'] & F\left(N\right) \arrow[r, "F(q)"] \arrow[d, "\beta'"'] & F\left(\mathrm{Coeq}(f,g)\right) \arrow[dl, "F(v)"'] \arrow[ddl, dotted, "v"] \\
 & F(A') \arrow[u, "\alpha'"'] \arrow[ul, "F(u)"] & F(B') \arrow[d, "\psi^{-1}"'] & \\
 & A \arrow[u, "\phi^{-1}"'] \arrow[uul, dotted, "u"] & B &
\end{tikzcd}$$

If $u = F(u')\phi^{-1}$ and $v = \psi^{-1}F(v')$, then we have that

$$F(e)u = F(e)(F(u')\phi^{-1}) = F(eu')\phi^{-1} = F(\alpha'')\phi^{-1} = \alpha'\phi^{-1} = \alpha\phi\phi^{-1} = \alpha,$$

and

$$vF(q) = \psi^{-1}F(v')F(q) = \psi^{-1}F(v'q) = \psi^{-1}F(\beta'') = \psi^{-1}\beta' = \psi^{-1}\psi\beta = \beta.$$

Therefore, we conclude that

$$F\left(\mathrm{Eq}(f,g)\right) \cong \mathrm{Eq}(F(f),F(g)) \quad \text{and} \quad F\left(\mathrm{Coeq}(f,g)\right) \cong \mathrm{Coeq}(F(f),F(g)).$$

□

Lemma 2.10. *Let $F : R-\mathrm{Mod} \to S-\mathrm{Mod}$ be an equivalence of categories. If $f,g : A \to B$ are homomorphisms in $R-\mathrm{Mod}$, then*

$$F(f+g) = F(f) + F(g)$$

Proof. First we have that

$$\begin{tikzcd}
A \arrow[r, "f+g"] \arrow[d, "\Delta_A"'] & B \\
A \oplus A \arrow[r, "f\oplus g"'] & B \oplus B \arrow[u, "\nabla_B"']
\end{tikzcd}$$

where $\Delta_A(a) = (a,a)$ and $\nabla_B(b_1,b_2) = b_1 + b_2$.

Let $\rho_i : A \oplus A \to A$ and $\pi_i : B \oplus B \to B$ be the natural projections to the $i-$th coordinate. From Proposition 2.7, we have that

$$\mathrm{Eq}(\rho_1,\rho_2) = \{(a,a) \in A \oplus A \mid a \in A\} \cong A.$$

Similarly, using Proposition 2.8, we have that

$$\begin{aligned}
(b_1,b_2) + \mathrm{Img}[\pi_2 \oplus (-\pi_2)] &= (b_1,b_2) + (b_2,-b_2) + \mathrm{Img}[\pi_2 \oplus (-\pi_2)] \\
&= (b_1+b_2,0) + \mathrm{Img}[\pi_2 \oplus (-\pi_2)],
\end{aligned}$$

since $\pi_2 \oplus (-\pi_2) = \pi_2 \oplus 0 - 0 \oplus \pi_2$. It follows that $(\mathrm{Coeq}(\pi_2 \oplus 0, 0 \oplus \pi_2)) \cong (B, \nabla_B)$.

Applying the functor F we have the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f+g)"] \arrow[d, "F(\Delta_A)"'] & F(B) \\
F(A) \oplus F(A) \arrow[r, "F(f)\oplus F(g)"'] & F(B) \oplus F(B) \arrow[u, "F(\nabla_B)"']
\end{tikzcd}$$

Using Proposition 2.9, we have that $F(\Delta_A) = \Delta_{F(A)}$ and $F(\nabla_B) = \nabla_{F(B)}$. Thus, we conclude that

$$F(f+g) = F(f) + F(g).$$

□

Definition 2.11. Two rings R and S are *(right) Morita equivalent* if the categories $\mathrm{Mod}-R$ and $\mathrm{Mod}-S$ are equivalent. We denote this by $R \simeq S$.

One of the goals of the following section is to show that $\mathrm{Mod}-R$ and $\mathrm{Mod}-S$ are equivalent if and only if $R-\mathrm{Mod}$ and $S-\mathrm{Mod}$ are equivalent.

Lemma 2.12 (Splitting Lemma). *In an Abelian category, let*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

be an exact sequence. The following statements are equivalent:

(1) *There is an isomorphism $B \overset{\psi}{\cong} A \oplus C$ such that $\iota_A = \psi\alpha$ is the natural inclusion of A into the direct sum, and $\rho_C = \beta\psi^{-1}$ is the natural projection from the direct sum into C.*
(2) *There is an homomorphism $g : C \to B$ such that $\beta g = 1_B$.*
(3) *There is an homomorphism $f : B \to A$ such that $f\alpha = 1_A$.*

If a short exact sequence satisfies the above properties, we say that the sequence and the morphisms α and β split.

Proof.
[(1) $\implies$ (2)]. Let $\iota_C : C\ toA \oplus C$ the natural inclusion. If we define $g = \psi^{-1}\iota_C$, then we have that

$$\beta g = \beta\psi^{-1}\iota_C = \rho_C\iota_C = 1_C,$$

which proves the statement.

[(2) $\implies$ (3)]. First, using that $\beta g = 1_B$, note that for any $b \in B$

$$\beta(b - g\beta(b)) = \beta(b) - (\beta g)\beta(b) = \beta(b) - \beta(b) = 0.$$

Using that the sequence is exact, we have that $b - g\beta(b) \in \ker(\beta) = \mathrm{Img}(\alpha)$. Now, using that α is injective, we have that $A \overset{\alpha}{\cong} \mathrm{Img}(\alpha)$. Define $f(b) = \alpha^{-1}(b - g\beta(b))$. Note that

$$f\alpha(a) = f(\alpha(a)) = \alpha^{-1}(\alpha(a) - g\beta(\alpha(a))) = \alpha^{-1}(\alpha(a)) = a.$$

Thus, we proved the statement.

[(3) $\implies$ (1)]. First, using that $f\alpha = 1_A$, we have that for any $b \in B$

$$f(b - \alpha f(b)) = f(b) - (f\alpha)f(b) = 0.$$

Since $b = b - \alpha f(b) + \alpha f(b)$, we have that $B = \ker(f) + \operatorname{Img}(\alpha)$. Now, if $b \in \ker(f) \cap \operatorname{Img}(\alpha)$, then $f(b) = 0$ and there is $a \in A$ such that

$$b = \alpha(a) = \alpha((f\alpha)(a)) = \alpha f(\alpha(a)) = \alpha f(b) = 0.$$

It follows that $B = \ker(f) \oplus \operatorname{Img}(\alpha)$. Using that β is surjective and that $\operatorname{Img}(\alpha) = \ker(\beta)$, we have that

$$C \cong \frac{B}{\ker(f)} = \frac{\ker(f) \oplus \operatorname{Img}(\alpha)}{\ker(\beta)} \cong \frac{\ker(f) \oplus \ker(\beta)}{\ker(\beta)} \cong \ker(f).$$

Since α is injective, we have that $A \cong \operatorname{Img}(\alpha)$, and so

$$B = \ker(f) \oplus \operatorname{Img}(\alpha) \cong C \oplus A \cong A \oplus C.$$

□

DEFINITION 2.13. Let G and M be modules over a ring R. We say that G *generates* M if and only if there is a nonempty set A and an epimorphism

$$G^A \to M \to 0.$$

Furthermore, if G generates every $R-$module, we say that G is a *generator*.

LEMMA 2.14. *Let G and M be $R-$modules. If G generates M, then there is a subset $H \subseteq \operatorname{Hom}_R(G, M)$ such that*

$$M = \sum_{h \in H} h(G).$$

PROOF. Using that G generates M, there is a set A and an epimorphism $G^A \xrightarrow{\varphi} M$. Consider the following commutative diagram

$$\begin{tikzcd} G^A \arrow[rr, two heads, "\varphi"] & & M \\ & G \arrow[ul, "\iota_\alpha"] \arrow[ur, "\varphi_\alpha"'] & \end{tikzcd}$$

If $(g_\alpha)_{\alpha \in A} \in G^A$, then

$$(g_\alpha)_{\alpha \in A} = \sum_{\alpha \in A} \iota_\alpha(g_\alpha),$$

because $g_\alpha = 0$ but finitely many $\alpha \in A$. Therefore,

$$M = \varphi\left(G^A\right) \subseteq \sum_{\alpha \in A} \varphi_\alpha(G) \subseteq M,$$

and so, with $H = \{\varphi_\alpha\}_{\alpha \in A}$ we have the desired result.

□

PROPOSITION 2.15. *For any $R-$module G, the following statements are equivalent:*

(1) G is a generator.

(2) There exists $n \in \mathbb{N}$ and an $R-$module G', such that

$$G^n \cong R \oplus G'$$

PROOF.

[(1) $\implies$ (2)]. Using the definition of a generator, there is a nonempty set A and an epimorphism

$$G^A \xrightarrow{\rho} R \to 0.$$

Thus, there is $(g_\alpha)_{\alpha\in A} \in G^A$ such that $\rho\left((g_\alpha)_{\alpha\in A}\right) = 1$. Using that $G^{(A)}$ is a direct sum, we have that there are $\alpha_1, \ldots, \alpha_n \in A$ such that $g_{\alpha_i} \neq 0$ and $g_\alpha = 0$, for $\alpha \neq \alpha_i$. Therefore, we can take π as the restriction

$$\bigoplus_{i=1}^{n} G_{\alpha_i} \to G^{(A)} \to R$$

and it still is an epimorphism. Define $h : R \to \bigoplus_{i=1}^{n} G_{\alpha_i}$ by

$$h(r) = r\sum_{i=1}^{n} g_{\alpha_i}.$$

Since $\pi h(r) = r$ for all $r \in R$, by splitting lemma 2.12, we have that π splits, and so

$$G^n \cong \bigoplus_{i=1}^{n} G_{\alpha_i} \cong R \oplus \ker(pi).$$

Thus, we proved the statement.

[(2) $\implies$ (1)]. Note there is an epimorphism $G \to R$. Let M be any $R-$module. Since R is a generator, there is a nonempty set A and an epimorphism $R^{(A)} \to A$. It follows that

$$G^{(A)} \to R^{(A)} \to A$$

is an epimorphism. $\square$

DEFINITION 2.16. Let P be an module over a ring R. We say that P is *projective* if for every epimorphism $f : M \to N$ and every homomorphism $g : P \to N$, there is a unique $h : P \to M$ such that $fh = g$, i.e., the diagram commutes

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "h"'] \arrow[d, "g"] & \\
M \arrow[r, two heads, "f"'] & N \arrow[r] & 0
\end{tikzcd}$$

LEMMA 2.17. *Every free module is projective.*

PROOF. Given a set A, consider $P = R^{(A)}$ be the free module over A. Let $f : M \to N$ be an epimorphism, and $g : P \to N$ any morphism. For every $\alpha \in A$, write $1_\alpha \in P$ the unit corresponding to R_α in P. Now, using that f is epi, there are $m_\alpha \in M$ such that

$$g(1_\alpha) = f(m_\alpha).$$

Define $h : P \to M$ by setting $h(1_\alpha) = m_\alpha$, and extending by linearity. Thus, by construction, we have that $hf = g$, i.e., P is projective. $\square$

PROPOSITION 2.18. *Given a $R-$module P, the following statements:*

(1) P is projective.
(2) Every epimorphism $M \to P \to 0$ splits.

(3) P is isomorphic to a direct summand of a free module, i.e., there is a set A and a P' module such that

$$R^{(A)} \cong P \oplus P'.$$

PROOF.

[(1) $\implies$ (2)] Let $f : M \to P$ be any epimorphism. Using that P is projective, we have that there is $g : P \to M$ such that $gf = 1_P$. By splitting lemma 2.12, we have that the epimorphism f splits.

[(2) $\implies$ (3).] Since R is a generator, there is a set A and a epimorphism $f : R^{(A)} \to P$. By hypothesis, the epimorphism f splits, specifically, if $P' = \ker(f)$, then $R^{(A)} \cong P \oplus P'$.

[(3) $\implies$ (1).] Let $f : M \to N$ be any epimorphism, and $g : P \to N$ any morphism. Since P is isomorphic to a direct summand of a free module F, we have that the identity in P factors through F, which is a projective module. It follows the following commutative diagram

$$\begin{tikzcd}
 & & P \arrow[dl] \arrow[d, "1_P"] & \\
 & F \arrow[r] \arrow[dl, dashed] \arrow[dr] & P \arrow[d] & \\
M \arrow[rr] & & N \arrow[r] & 0
\end{tikzcd}$$

Thus, P is projective. $\square$

COROLLARY 2.19. *An $R-$module P is a finitely generated* projective *module if and only if there is $n \in \mathbb{N}$ and a module P', such that*

$$R^{\oplus n} \cong P \oplus P'.$$

DEFINITION 2.20. An $R-$module P is a *progenerator* if it is a finite projective generator.

DEFINITION 2.21. Let M be a right $S-$module. Define $R = \mathrm{End}_S(M)$. Note that M is a left $R-$module by the action

$$f \cdot m = f(m), f \in R \text{ and } m \in M.$$

Furthermore, since $f \in \mathrm{End}_S(M)$, we have that for any $s \in S$

$$f \cdot (m \cdot s) = f(m \cdot s) = f(m) \cdot s = (f \cdot m) \cdot s$$

and so M is an $(R - S)-$bimodule. We denote by $\mathrm{BiEnd}_S(M) = \mathrm{End}_R(M)$, and we call it the *biendomorphism ring*.

REMARK. For any right $S-$module M as before, there is a morphism $S \to \mathrm{BiEnd}_S(M)$ given by the map $s \mapsto - \cdot s$.

PROPOSITION 2.22. *Let M, M', and M'' be right $S-$modules.*

- *If $M = M' \oplus M''$, and M' generates M'', then there is a monomorphism*

$$\mathrm{BiEnd}_S(M) \xrightarrow{\mathrm{Res}} \mathrm{BiEnd}_S(M').$$

corresponding to the restriction map.

- $\mathrm{BiEnd}_S(M) \cong \mathrm{BiEnd}_S(M^{\oplus n})$*, for any $n \in \mathbb{N}$.*

Proof.

- Let $R = \mathrm{End}_S(M)$ and $R' = \mathrm{End}_S(M')$. If $\iota : M' \to M$ and $\rho : M \to M'$ are the inclusion and the projection for the direct summand M', then for any $f \in R$, we have that $\rho f \iota \in R'$. Similarly, if $f' \in \mathrm{End}_S(M') = R'$, then we have that $\iota f' \rho \in R$. Therefore, we have that
$$\rho R \iota \subseteq R' = \rho\iota R' \rho\iota \subseteq \rho R \iota,$$
because $1_{M'} = \rho\iota$.

  Let $\psi \in \mathrm{End}_R(M) = \mathrm{BiEnd}_S(M)$. If $\mathrm{Res}(\psi) = \rho\psi\iota$, then we have that $\mathrm{Res}(\psi)$ is an $\mathbb{Z}-$homomorphism because all three are. We need to show that is $R'-$linear. Let $f' \in R'$. Using that $f = \iota f' \rho \in R$, we have that
$$\begin{aligned}\mathrm{Res}(\psi) f' &= \rho\psi\iota f' \\ &= \rho\psi(\iota f'\rho)\iota \\ &= \rho(\psi f)\iota \\ &= \rho(f\psi)\iota \\ &= \rho\iota f'(\rho\psi\iota) \\ &= f' \,\mathrm{Res}(\psi).\end{aligned}$$
It follows that $\mathrm{Res}(\psi) \in \mathrm{End}_{R'}(M') = \mathrm{BiEnd}_S(M')$. Now, we need to show that Res is an homomorphism. If $\psi_1, \psi_2 \in \mathrm{BiEnd}_S(M)$, then
$$\mathrm{Res}(\psi_1 + \psi_2) = \rho(\psi_1 + \psi_2)\iota = \rho\psi_1\iota + \rho\psi_2\iota.$$
If $\varphi = \iota\rho \in R$, then
$$\begin{aligned}\mathrm{Res}(\psi_1\psi_2) = \rho\psi_1\psi_2\iota &= \rho\psi_1\psi_2\iota\rho\iota \\ &= \rho\psi_1\psi_2\varphi\iota \\ &= \rho\psi_1\varphi\psi_1\iota \\ &= \rho\psi_1\iota\rho\psi_1\iota \\ &= \mathrm{Res}(\psi_1)\,\mathrm{Res}(\psi_2).\end{aligned}$$
Thus, we conclude that in fact $\mathrm{Res} : \mathrm{BiEnd}_S(M) \to \mathrm{BiEnd}_S(M')$ is a ring homomorphism.

  Now, let $\psi \in \mathrm{BiEnd}_S(M)$ such that $\mathrm{Res}(\psi) = 0$. It follows that
$$0 = \iota\,\mathrm{Res}(\psi) = \iota\rho\psi\iota = \varphi\psi\iota = \psi\varphi\iota = \psi\iota\rho\iota = \psi\iota$$
If we have that M' generates M'', then M' also generates M. Using Lemma 2.14, if $H = \mathrm{Hom}_S(M', M)$, then
$$\sum_{h\in H} h(M') = M$$
  Also note that

$$H \subseteq \mathrm{Hom}_S(M', M) \oplus \mathrm{Hom}_S(M'', M) = \mathrm{Hom}_S(M' \oplus M'', M) = \mathrm{Hom}_S(M, M) = R,$$

with the inclusion given by $\mathrm{Hom}_S(\iota, M) : \mathrm{Hom}_S(M', M) \to \mathrm{Hom}_S(M, M)$. Therefore, we have that $M = R\iota(M')$, and so

$$\psi(M) = \psi(R\iota(M')) = R\psi\iota(M') = R \cdot 0 = 0.$$

With this, we conclude that Res is an injective homomorphism of rings.

- As shown before, the map $\mathrm{Res} : \mathrm{BiEnd}_S(M^n) \to \mathrm{BiEnd}_S(M)$ is an injective ring homomorphism. We only need to show that Res is surjective.

  Let $\iota_\ell : M \to M^n$ be the inclusion in the $\ell-$coordinate, $\rho_k : M^n \to M$ the projection of the $k-$coordinate, $R = \mathrm{End}_S(M^n)$ and $R' = \mathrm{End}_S(M)$. We have that

$$\rho_k R \iota_\ell \subseteq \rho_\ell R \iota_\ell.$$

  Let $\psi \in \mathrm{End}_{R'}(M)$. Note that

$$\begin{aligned}\psi^n(m_1, \ldots, m_n) &= (\psi(m_1), \ldots, \psi(m_n)) \\ &= \sum_{\ell=1}^{n} \iota_\ell(\psi(m_\ell)) \\ &= \sum_{\ell=1}^{n} \iota_\ell \psi(m_\ell) \\ &= \sum_{\ell=1}^{n} \iota_\ell \psi(\rho_\ell(m_1, \ldots, m_n)).\end{aligned}$$

  Using that $\rho_k \iota_\ell = \delta_{k\ell} \mathbb{1}_M$, we have that $\psi^n \iota_\ell = \iota_\ell \psi$, and $\rho_k \psi^{(n)} = \psi \rho_k$. Recall that $\rho_k f \iota_\ell \in R'$ and $\psi \in \mathrm{End}_{R'}(M)$, thus

$$\begin{aligned}\rho_k f \psi^n \iota_\ell &= (\rho_k f \iota_\ell)\psi \\ &= \psi(\rho_k f \iota_\ell) \\ &= \rho_k \psi^n f \iota_\ell.\end{aligned}$$

  By uniqueness of the universal property of the direct product, we have that

$$f \psi^n \iota_\ell = \psi^n f \iota_\ell.$$

  By uniqueness of universal property of the direct sum, we conclude that

$$\psi^n f = f \psi^n,$$

  and so $\psi^n \in \mathrm{End}_R(M^n)$. Since $\mathrm{Res}\, \psi^n = \rho_1 \psi^{(n)} \iota_1 = \psi$, we have that Res is surjective, which concludes the proof.

□

THEOREM 2.23. *If P is a progenerator right $S-$module, and $R \cong \mathrm{End}_S(P)$, then, P is also a progenerator as a left $R-$module.*

PROOF. As we showed before, there are $n, m \in \mathbb{N}$ and S', P' right $S-$modules such that

$$P^{\oplus m} \cong S \oplus S', \quad \text{and} \quad S^{\oplus n} \cong P \oplus P'.$$

It follows that

$$\begin{aligned} P^{\oplus n} &\cong \mathrm{Hom}_S\left(S^{\oplus n}, P\right) \\ &\cong \mathrm{Hom}_S\left(P \oplus P', P\right) \\ &\cong R \oplus \mathrm{Hom}_S\left(P', P\right), \end{aligned}$$

which shows that P is a generator as an $R-$module. Similarly have that

$$\begin{aligned} R^{\oplus m} &\cong \mathrm{Hom}_S\left(P^{\oplus m}, P\right) \\ &\cong \mathrm{Hom}_S\left(S \oplus S', P\right) \\ &\cong P \oplus \mathrm{Hom}_S\left(S', P\right), \end{aligned}$$

and so P is a finitely generated projective $R-$module. $\square$

2. Morita's Theorem

Our goal in this section is to prove Morita's Theorem, and using it define the Frobenius Descent.

THEOREM 2.24 (Morita's Theorem). *Let R and S be rings. Then two additive functors $F : \mathrm{Mod}-R \to \mathrm{Mod}-S$ and $G : \mathrm{Mod}-S \to \mathrm{Mod}-R$ are inverse equivalences if and only if here exists a $(R, S)-$bimodule P such that:*

(1) P is a progenerator in both $R-\mathrm{Mod}$ and $\mathrm{Mod}-S$.
(2) $R \cong \mathrm{End}_S(P)$ and $S \cong \mathrm{End}_R(P)$.
(3) $F \cong - \otimes_R P$, and $G \cong \mathrm{Hom}_S(P, -)$.

Furthermore, if there is such a $(R, S)-$bimodule P satisfying these conditions, then

$$G \cong - \otimes_S \mathrm{Hom}_R(P, R)$$

LEMMA 2.25. *If R and S are Morita equivalent through $F : \mathrm{Mod}-R \to \mathrm{Mod}-S$ and $G : \mathrm{Mod}-S \to \mathrm{Mod}-R$, then there are $\mathbb{Z}-$isomorphisms*

$$\mathrm{Hom}_S(F(M), N) \to \mathrm{Hom}_R(M, G(N))$$

and

$$\mathrm{Hom}_S(N, F(M)) \to \mathrm{Hom}_R(G(N), M).$$

PROOF. Using theorem 2.3, any equivalence of categories is full faithful and essentially surjective and so we have the isomorphism of sets

$$\mathrm{Hom}_S(F(M), N) \xrightarrow{\cong} \mathrm{Hom}_R(G(F(M)), G(N)).$$

Using the lemma 2.10, we have that the map is an homomorphism of Abelian groups. Now, since $GF(M) \cong M$ as $R-$modules, we have that

$$\mathrm{Hom}_R(G(F(M)), G(N)) \cong \mathrm{Hom}_R(M, G(N))$$

as $R-$modules, and so, we have that

$$\mathrm{Hom}_S(F(M), N) \cong \mathrm{Hom}_R(M, G(N)).$$

By the same reasoning, we conclude that $\mathrm{Hom}_S(N, F(M)) \to \mathrm{Hom}_R(G(N), M)$ is $\mathbb{Z}-$linear. $\square$

LEMMA 2.26. *Let P be a right $S-$module, M a right $R-$module, and U a $(R,S)-$bimodule. Then there is a homomorphism*

$$M \otimes_R \operatorname{Hom}_S(P,U) \xrightarrow{\eta} \operatorname{Hom}_S(P, M \otimes_R U)$$

defined by $[\eta(m \otimes \delta)](p) = m \otimes \delta(p)$. If P is a finitely generated and projective $S-$module, then η is an isomorphism.

PROOF. From the definition η is a $\mathbb{Z}-$homomorphism. If $g : M \to M'$, $f : P' \to P$, and $h : U \to U'$ are maps in the respective categories, then we have two maps

$$\begin{aligned} M \otimes_R \operatorname{Hom}_S(P,U) &\to M' \otimes_R \operatorname{Hom}_S(P',U') \\ m \otimes \varphi &\mapsto g(m) \otimes h\varphi f, \end{aligned}$$

and

$$\begin{aligned} \operatorname{Hom}_S(P, M \otimes_R U) &\to \operatorname{Hom}_S(P', M' \otimes_R U') \\ \psi &\mapsto [g \otimes h]\psi f, \end{aligned}$$

given by the naturality of both hom and tensor functors. Let $p' \in P'$, $m \in M$, and $\varphi \in \operatorname{Hom}_S(P,U)$, then

$$\begin{aligned} [g \otimes h][\eta(m \otimes \varphi)]f(p') &= [g \otimes h][m \otimes \varphi(f(p'))] \\ &= g(m) \otimes h\varphi f(p') \\ &= [\eta[g(m) \otimes h\varphi f]](p'). \end{aligned}$$

This shows that η is natural in all three entries. For convenience, we write

$$F(P) = M \otimes_R \operatorname{Hom}_S(P,U), \quad \text{and} \quad G(P) = \operatorname{Hom}_S(P, M \otimes_R U).$$

Thus, η becomes a natural transformation between F and G. When $P = S$, we have that the results follows from

$$F(S) = M \otimes_R \operatorname{Hom}_S(S,U) \cong M \otimes_R U \cong \operatorname{Hom}_S(S, M \otimes_R U) = G(S)$$

Using the additivity from both the *hom* and *tensor* functors, we have that both F and G are additive. Thus, for any $n \in \mathbb{N}$,

$$F(S^{(n)}) \cong G(S^{(n)}).$$

Now, let P be any finitely generated and projective $S-$module. This means that there is $n \in \mathbb{N}$ and a $S-$module P' such that $S^{(n)} \cong P \otimes P'$. We have the split exact sequence

$$0 \to P \to S^{(n)} \to P' \to 0,$$

which induces the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F(P) \arrow[r] \arrow[d, "\eta"] & F(S^{(n)}) \arrow[r] \arrow[d, "\eta"] & F(P') \arrow[r] \arrow[d, "\eta"] & 0 \\ 0 \arrow[r] & G(P) \arrow[r] & G(S^{(n)}) \arrow[r] & G(P') \arrow[r] & 0. \end{tikzcd}$$

By the *Five Lemma*, we conclude that $F(P) \cong G(P)$. $\square$

Lemma 2.27. *Let P be a left $R-$module, N a right $S-$module, and U a $(R,S)-$bimodule. Then there is a homomorphism*

$$\mathrm{Hom}_S(U,N)\otimes_R P \xrightarrow{\nu} \mathrm{Hom}_S(\mathrm{Hom}_R(P,U),N)$$

given by $[\nu(\gamma\otimes p)](\delta)=\gamma\delta(p)$. If P is a finitely generated and projective $R-$module, then ν is an isomorphism.

Proof. Similarly as before, we have that ν is a $\mathbb{Z}-$homomorphism that is natural in all three entries. For convenience, we write $F(P)=\mathrm{Hom}_S(U,N)\otimes_R P$ and $\mathrm{Hom}_S(\mathrm{Hom}_R(P,U),N)$, and so ν becomes a natural transformation between F and G. When $P=R$, we have that the results follow from

$$F(R)=\mathrm{Hom}_S(U,N)\otimes_R R\cong \mathrm{Hom}_S(U,N)\cong \mathrm{Hom}_S(\mathrm{Hom}_R(R,U),N)=G(R)$$

Using the additivity from both the hom and tensor functors, we have that both F and G are additive. Thus, for any $n\in\mathbb{N}$,

$$F(R^{(n)})\cong G(R^{(n)}).$$

Now, let P be any finitely generated and projective $R-$module. This means that there is $n\in\mathbb{N}$ and a $S-$module P' such that $R^{(n)}\cong P\otimes P'$. So we have the split exact sequence

$$0\to P\to R^{(n)}\to P'\to 0,$$

which induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F(P) \arrow[r] \arrow[d, "\nu"] & F(R^{(n)}) \arrow[r] \arrow[d, "\nu"] & F(P') \arrow[r] \arrow[d, "\nu"] & 0 \\
0 \arrow[r] & G(P) \arrow[r] & G(R^{(n)}) \arrow[r] & G(P') \arrow[r] & 0.
\end{tikzcd}$$

By the Five lemma, we conclude that $F(P)\cong G(P)$. □

Proposition 2.28. *If P is an $(R,S)-$bimodule that is a progenerator as an* $\mathrm{Mod}-S$ *and $R\cong \mathrm{End}_S(P)$. Then, P a progenerator in* $\mathrm{Mod}-R$ *and $S\cong \mathrm{End}_R(P)$.*

Proof. The first part of the result is shown in Theorem 2.23. We only need to show that $S\cong \mathrm{Hom}_R(P,P)$. In order to prove that, we use the Proposition 2.22 and we have the commutative diagram

$$\begin{tikzcd}
 & S \arrow[r, "\cong"] \arrow[dl, hook] \arrow[d, hook] \arrow[dr, hook] & \mathrm{BiEnd}_S(S) \\
\mathrm{BiEnd}_S(P) \arrow[r, "\cong"] & \mathrm{BiEnd}_S(P^{\oplus m}) \arrow[r, "\cong"] & \mathrm{BiEnd}_S(S\oplus S') \arrow[u, hook]
\end{tikzcd}$$

Thus, we have a monomorphism from $\mathrm{BiEnd}_S(P)\cong \mathrm{End}_R(P)$ into S, which itself injects into $\mathrm{End}_R(P)$, and so, we conclude that $\mathrm{End}_R(P)\cong S$. □

Proof of Morita's Theorem 2.24. First, suppose that F and G are inverse equivalences. Let $P=F(R)$, which by definition is a right $S-$module. We have that P is a $(R,S)-$bimodule, since

$$\mathrm{End}_S(P)=\mathrm{Hom}_S(F(R),F(R))\cong \mathrm{Hom}_R(R,R)\cong R$$

acts on the left, and commutes with the right S action.

We want to show that P is a progenerator as a right $S-$module. Let N be a right $S-$module. Using the Theorem 2.3, there is an $R-$module M with $F(M) \cong N$. Recall that R is a generator. Then, there is a nonempty set A and an epimorphism $R^A \to M \to 0$. It follows that

$$P^A \cong F(R^A) \to F(M) \cong N \to 0.$$

Since N was arbitrary, we conclude that P is a generator of $S-$modules.

Now, consider an epimorphism $N \to P \to 0$. Recall that $G(P) = G(F(R)) \cong R$. Using that R is projective and the Proposition 2.18, the epimorphism

$$G(N) \to R \to 0$$

splits. It follows that the epimorphism $F(G(N)) \cong N \to P \to 0$ splits, and so P is projective. Thus, the first condition is satisfied. Note that the second condition on P is satisfied by Proposition 2.28.

Now, using Lemma 2.25, we have that for any $S-$module N

$$G(N) \cong \mathrm{Hom}_R\left(R, G(N)\right) \cong \mathrm{Hom}_S\left(F(R), N\right) = \mathrm{Hom}_S\left(P, N\right)$$

In particular, using that $S \cong \mathrm{End}_R\left(P\right)$ and $R \cong \mathrm{End}_S\left(P\right)$, we have that

$$G(S) \cong \mathrm{Hom}_S\left(P, S\right) \cong \mathrm{Hom}_S\left(P, \mathrm{Hom}_R\left(P, P\right)\right) \cong \mathrm{Hom}_R\left(P, \mathrm{Hom}_S\left(P, P\right)\right) \cong \mathrm{Hom}_R\left(P, R\right)$$

Using lemma 2.27, it follows that for any $R-$module M,

$$\begin{aligned} F(M) &\cong \mathrm{Hom}_S\left(S, F(M)\right) \\ &\cong \mathrm{Hom}_R\left(G(S), M\right) \\ &\cong \mathrm{Hom}_R\left(\mathrm{Hom}_R\left(P, R\right), M\right) \\ &\cong \mathrm{Hom}_R\left(R, M\right) \otimes_R P \\ &\cong M \otimes_R P. \end{aligned}$$

Thus, the third condition is satisfied.

Now, suppose that such a bimodule P exists. Using the Lemmas 2.27 and 2.26, we have that

$$FG(N) \cong \mathrm{Hom}_S\left(P, M \otimes_R P\right) \cong M \otimes_R \mathrm{Hom}_S\left(P, P\right) \cong M,$$

and

$$GF(M) \cong \mathrm{Hom}_S\left(P, N\right) \otimes_R P \cong \mathrm{Hom}_S\left(\mathrm{Hom}_R\left(P, P\right), N\right) \cong N.$$

So, we have that F and G are inverse of each other.

For the last remark, we proceed similarly by defining $Q = G(S) \cong \mathrm{Hom}_R\left(P, R\right)$. Using that G is an equivalence of categories, we conclude that

$$G \cong - \otimes_S Q \cong - \otimes_S \mathrm{Hom}_R\left(P, R\right).$$

□

COROLLARY 2.29. *Let R and S be rings. Then* $\mathrm{Mod}-R \cong \mathrm{Mod}-S$ *if and only if* $R - \mathrm{Mod} \cong S - \mathrm{Mod}$. *Furthermore, there is $(S,R)-$bimodule P such that*

(1) P is a progenerator in both $S - \mathrm{Mod}$ *and* $\mathrm{Mod}-R$.
(2) $R \cong \mathrm{End}_S\left(P\right)$ and $S \cong \mathrm{End}_R\left(P\right)$.
(3) $F \cong P \otimes_R -$ and $G \cong \mathrm{Hom}_S\left(P, -\right) \cong \mathrm{Hom}_R\left(P, R\right) \otimes_S -$.

PROOF. Suppose that $R - \mathrm{Mod} \cong S - \mathrm{Mod}$. Then, $\mathrm{Mod} - R^{op} \cong \mathrm{Mod} - S^{op}$. Using Morita's Theorem 2.24, there is a $(R^{op}, S^{op})-$bimodule P such that is a progenerator in

$$R \cong R^{op} \cong \mathrm{End}_{S^{op}}(P) \cong \mathrm{End}_S(P), \quad \text{and} \quad S \cong S^{op} \cong \mathrm{End}_{R^{op}}(P) \cong \mathrm{End}_R(P),$$

and there are functors

$$F(-) = -\otimes_{R^{op}} P \cong P \otimes_R -, \quad \text{and} \quad G(-) = -\otimes_{S^{op}} \mathrm{Hom}_{R^{op}}(P, R^{op}) \cong \mathrm{Hom}_R(P,R) \otimes_S -,$$

which are inverse of each other from $R-\mathrm{Mod} \cong \mathrm{Mod} - R^{op}$ to $S-\mathrm{Mod} \cong \mathrm{Mod} - S^{op}$. Since P is a $(S,R)-$bimodule, we can take the functors $- \otimes_S P$ and $- \otimes_R \mathrm{Hom}_R(P,R)$, which are equivalences between $\mathrm{Mod} - S$ and $\mathrm{Mod} - R$, by Morita's Theorem. The converse is proven in a similar fashion. □

3. Morita equivalence: Ring and $n \times n-$Matrix ring

In this section we show that R and $S = \mathrm{Mat}_n(R)$, the ring of $n \times n$ matrices with entries in R, are Morita equivalent.

THEOREM 2.30. *Let M and M' be $R-$modules and $f : M \to M'$ a homomorphism of $R-$modules. We define the functor $F : R - \mathrm{Mod} \to S - \mathrm{Mod}$ by*

$$F(-) = R^n \otimes_R -,$$

and its inverse is

$$G(-) = \mathrm{Hom}_R(R^n, R) \otimes_S -$$

Let $P = R^n \cong R^n \otimes_R R$. It follows that P is a progenerator in $R - \mathrm{Mod}$. Using that

$$S \cong \mathrm{End}_R(R^n) \cong R^{n^2} \cong P^n \cong P \oplus R^{n^2-n}$$

we have that P is a generator and a finitely generated projective module, i.e., P is a progenerator in $S - \mathrm{Mod}$.

We only need to check that $R \cong \mathrm{End}_S(R^n)$. First, since there is an inclusion

$$\begin{aligned} R &\to S \\ r &\mapsto r\mathbb{1}_n, \end{aligned}$$

we have that

$$\mathrm{End}_S(R^n) \subseteq \mathrm{End}_R(R^n) \cong \mathrm{Mat}_n(R).$$

Let $\{e_i\}$ denote the canonical basis of R^n. For simplicity we consider $\{e_i\}$ as $n \times 1-$matrices (column vectors). $\{E_{ij}\}$ the $n \times n-$matrix with 1 in position (i,j) and 0 elsewhere. Let v be a $1 \times n-$matrix (a row vector) and w be a $n \times 1-$matrix (a column vector). We have that $e_i v$ is the $n \times n-$matrix with v in the $i-$th row and 0 elsewhere,

$$(e_i v)_{jk} = (e_i)_j v_k = \delta_{ij} v_k,$$

and that $w e_j^T$ is the $n \times n-$matrix with w in the $j-$th column and 0 elsewhere

$$(w e_j^T) ki = w_k (e_j^T)_i = \delta_{ij} w_k.$$

Using this, we have that $E_{ij} = e_i e_j^T$

$$(e_i e_j^T)_{k\ell} = (e_i)_k (e_j^T)_\ell = \delta_{ik}\delta_{j\ell} = (E_{ij})_{k\ell}$$

If M is a $n \times n-$matrix, then $e_j^T M$ is the $j-$th row of M and Me_i is the $i-$th column of M, because

$$(e_j^T M)_{1k} = \sum_{i=1}^{n} (e_j^T)_i M_{ik} = M_{jk} \quad \text{and} \quad (Me_i)_{k1} = \sum_{j=1}^{n} M_{kj}(e_i)_j = M_{ki}.$$

Using that

$$(ME_{ij})_{k\ell} = \sum_{h=1}^{n} M_{kh}(E_{ij})_{h\ell} = \sum_{h=1}^{n} M_{kh}\delta_{ih}\delta_{j\ell} = \delta_{j\ell}M_{ki},$$

we have that $ME_{ij} = Me_ie_j^T$ is the matrix whose $j-$th column is the $i-$th column of M and 0 elsewhere. Similarly with

$$(E_{ij}M)_{k\ell} = \sum_{h=1}^{n} (E_{ij})_{kh}M_{h\ell} = \sum_{h=1}^{n} \delta_{ik}\delta_{jh}M_{h\ell} = \delta_{ik}M_{j\ell}$$

we have that $E_{ij}M = e_ie_j^T M$ is the matrix whose $i-$th row is the $j-$th row of M.

Now, let $M \in \operatorname{End}_S(R^n)$, and consider it as a matrix. Thus, we have that $ME_{ii} = E_{ii}M$. It follows that for any $i \neq k$

$$M_{ki} = \delta_{ii}M_{ki} = (ME_{ii})_{ki} = (E_{ii}M)_{ki} = \delta_{ik}M_{ii} = 0,$$

and so M is a diagonal matrix.

We need to show that $M = r\mathbb{1}_n$, for some $r \in R$. Note that

$$E_{ij}E_{ji} = e_ie_j^T e_je_i^T = e_i \left\| e_j \right\|^2 e_i^T = e_ie_i^T = E_{ii}.$$

Thus,

$$M_{11} = (ME_{11})_{11} = (ME_{1i}E_{i1})_{11} = (E_{1i}ME_{i1})_{11} = \sum_{j=1}^{n} (E_{1i})_{1j}(ME_{i1})_{j1} = (ME_{i1})_{i1} = M_{ii}.$$

Therefore, $M = M_{11}\mathbb{1}_n$. we conclude that $R \cong \operatorname{End}_S(R^n)$ by the inclusion $R \hookrightarrow S$ defined earlier.

Using Morita's Theorem 2.24, we conclude that $F(-) = R^n \otimes_R -$ is a equivalence of categories with the inverse functor $G(-) = \operatorname{Hom}_R(R^n, R) \otimes_S -$.

4. Morita equivalence: Frobenius Descent

In this section, we define the Frobenius Descent.

PROPOSITION 2.31. *Let R be a ring, M be an $R-$submodule, and $W \subseteq R$ a multiplicative subset. Then the $R-$linear map*

$$\psi : W^{-1}R \otimes_R M \to W^{-1}M$$
$$\frac{r}{w} \otimes m \mapsto \frac{rm}{w}$$

is an isomorphism.

PROOF. We need to show that ψ is bijective. If $\frac{r}{w} \otimes m$ such that $\frac{rm}{w} = 0$, then there is $w' \in W$ such that $w'rm = 0$. It follows that

$$\frac{r}{w} \otimes m = \frac{w'r}{w'w} \otimes m = w'r\frac{1}{w'w} \otimes m = \frac{1}{w'w} \otimes w'rm = \frac{1}{w'w} \otimes 0 = 0,$$

and so, ψ is injective. Now, for any $\frac{m}{w} \in W^{-1}M$ there is $\frac{1}{w} \otimes m$ such that $\psi(\frac{1}{w} \otimes m) = \frac{m}{w}$, and so ψ is surjective. Thus, we conclude that ψ is bijective. $\square$

COROLLARY 2.32. *Let R be a ring and $W \subseteq R$ a multiplicative closed subset. Then*

$$W^{-1}(-) \cong W^{-1}R \otimes_R -$$

is a functor between $R-$modules and $W^{-1}R-$modules.

PROOF. Consider ψ as in Proposition 2.31. Let $f : M \to N$ be an homomorphism of $R-$modules. If $\frac{m}{w} \in W^{-1}M$, then

$$W^{-1}f\left(\frac{m}{w}\right) = \psi(1_{W^{-1}R}\otimes f)\psi^{-1}\left(\frac{m}{w}\right) = \psi(1_{W^{-1}R}\otimes f)\left(\frac{1}{w} \otimes m\right) = \psi\left(\frac{1}{w} \otimes f(m)\right) = \frac{f(m)}{w}.$$

Thus, for any $\frac{r}{w'} \in W^{-1}R$,

$$W^{-1}f\left(\frac{r}{w'}\frac{m}{w}\right) = W^{-1}f\left(\frac{rm}{w'w}\right) = \frac{f(rm)}{w'w} = \frac{rf(m)}{w'w}\frac{r}{w'}W^{-1}f\left(\frac{m}{w}\right).$$

We conclude that $W^{-1}f$ is $W^{-1}R-$linear. $\square$

PROPOSITION 2.33. *Let R be a ring, and M, N be $R-$modules. If $W \subseteq R$ is a multiplicative closed subset, then*

$$W^{-1}M \otimes_{W^{-1}R} W^{-1}N \cong W^{-1}\left(M \otimes_R N\right).$$

PROOF. Using Proposition 2.31, we have that

$$\begin{aligned} W^{-1}M \otimes_{W^{-1}R} W^{-1}N &\cong \left(W^{-1}R \otimes_R M\right) \otimes_{W^{-1}R} \left(W^{-1}R \otimes_R N\right) \\ &\cong W^{-1}R \otimes_R \left(M \otimes_R N\right) \\ &\cong W^{-1}\left(M \otimes_R N\right). \end{aligned}$$

$\square$

PROPOSITION 2.34. [**8**, Tag 02C6] *Let $W_1, W_2 \subseteq R$ be multiplicative closed subsets, and M be an $R-$module. If $\overline{W_1}$ is the image of W_1 in $W_2^{-1}R$, then*

$$\left(W_1W_2\right)^{-1}R \cong \overline{W_1}^{-1}\left(W_2^{-1}R\right).$$

PROPOSITION 2.35. *Let $L \xrightarrow{f} M \xrightarrow{g} N$ be an exact sequence of $R-$modules. Then $W^{-1}L \to W^{-1}M \to W^{-1}N$ is also exact.*

PROOF. Using that localization is a functor, we have that $W^{-1}L \to W^{-1}M \to W^{-1}N$ is a complex. Let $\frac{m}{w} \in W^{-1}M$ such that $0 = \left(W^{-1}g\right)\left(\frac{m}{w}\right) = \frac{g(m)}{w}$. By definition, there is $w' \in W$ such that $0 = w'g(m) = g(w'm)$. It follows that $w'm \in \ker(g) = \operatorname{Img}(f)$. Thus, there is $l \in L$ such that $f(l) = w'm$, and so

$$W^{-1}f\left(\frac{l}{w'w}\right) = \frac{f(l)}{w'w} = \frac{w'm}{w'w} = \frac{m}{w}.$$

We conclude that the sequence $W^{-1}L \to W^{-1}M \to W^{-1}N$ is exact. $\square$

LEMMA 2.36. *Let R be a ring, M and N are $R-$modules, and $W \subseteq R$ a multiplicative closed subset. If M is finitely presented, then*

$$W^{-1}\left(\operatorname{Hom}_R\left(M, N\right)\right) \cong \operatorname{Hom}_{W^{-1}R}\left(W^{-1}M, W^{-1}N\right)$$

PROOF. Using that M is finitely presented, there are $n, m \in \mathbb{N}$ and an exact sequence

$$R^m \to R^n \to M \to 0. \tag{1}$$

For convenience, write $S = W^{-1}R$. Localize the sequence (1) to get the exact sequence

$$S^m \to S^n \to S \otimes_R M \to 0. \tag{2}$$

On the other hand, if we use the functor $\mathrm{Hom}_R(-, N)$ on the sequence (1), then we have the exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to N^n \to N^m. \tag{3}$$

On this sequence, we take the localization to get

$$0 \to S \otimes_R \mathrm{Hom}_R(M, N) \to (S \otimes_R N)^n \to (S \otimes_R N)^m.$$

Now, if we use the functor $\mathrm{Hom}_S(-, S \otimes_R N)$, then we have the exact sequence

$$0 \to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) \to (S \otimes_R N)^n \to (S \otimes_R N)^m.$$

Note there is an $S-$linear homomorphism

$$\begin{aligned} \varphi : S \otimes_R \mathrm{Hom}_R(M, N) &\to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) \\ 1 \otimes f &\mapsto \mu_1 \otimes f. \end{aligned}$$

Thus, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & S \otimes_R \mathrm{Hom}_R(M, N) & \longrightarrow & (S \otimes_R N)^n & \longrightarrow & (S \otimes_R N)^m \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) & \longrightarrow & (S \otimes_R N)^n & \longrightarrow & (S \otimes_R N)^m \end{array}.$$

By *Five Lemma*, we have that φ is an isomorphism, and so

$$\begin{aligned} W^{-1}(\mathrm{Hom}_R(M, N)) &\cong S \otimes_R \mathrm{Hom}_R(M, N) \\ &\cong \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) \\ &\cong \mathrm{Hom}_{W^{-1}R}\left(W^{-1}M, W^{-1}N\right). \end{aligned}$$

□

DEFINITION 2.37. Let R be a ring and M an $R-$module.

(1) We say that M is locally free if we can cover $\mathrm{Spec}(R)$ by $\{D(f_i)\}_{i \in \mathscr{A}}$, such that M_{f_i} is a free $R_{f_i}-$module, for all $i \in \mathscr{A}$.
(2) We say that M is finite locally free if we can choose the covering such that M_{f_i} is finitely generated free module.

LEMMA 2.38. [**8**, Lemma 00HN] *Let R be a ring, and M be an $R-$module.*

(1) Let M be an $R-$module. For $m \in M$ the following are equivalent:
  (a) $m = 0$.
  (b) m maps to zero in $M_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.
  (c) m maps to zero in $M_{\mathfrak{m}}$ for all $\mathfrak{m} \in \max R$.
(2) For an $R-$module M the following are equivalent:
  (a) $M = 0$.
  (b) $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

   (c) $M_{\mathfrak{m}} = 0$ *for all* $\mathfrak{m} \in \max R$.

(3) *Given a complex* $M_1 \to M_2 \to M_3$ *of* $R-$*modules the following are equivalent:*
   (a) $M_1 \to M_2 \to M_3$ *is exact.*
   (b) $(M_1)_{\mathfrak{p}} \to (M_2)_{\mathfrak{p}} \to (M_3)_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$.
   (c) $(M_1)_{\mathfrak{m}} \to (M_2)_{\mathfrak{m}} \to (M_3)_{\mathfrak{m}}$ *for all* $\mathfrak{m} \in \max R$.

LEMMA 2.39. [**8**, Tag 00EO] *Let* R *be a ring, and* M *be an* $R-$*module. Suppose there are* $f_1, \ldots, f_n \in R$*, for some* $n \in \mathbb{N}$*, such that* $(f_1, \ldots, f_n) = R$.

(1) *If each* $M_{f_i} = 0$*, then* $M = 0$.
(2) *If each* M_{f_i} *is finite* $R_{f_i}-$*module, then* M *is finite* $R-$*module.*
(3) *If each* M_{f_i} *is finitely presented* $R_{f_i}-$*module, then* M *is finitely presented* $R-$*module.*
(4) *Let* $M_1 \to M_2 \to M_3$ *be complex of* $R-$*modules. If each* $(M_1)_{f_i} \to (M_2)_{f_i} \to (M_3)_{f_i}$ *is exact, then* $M_1 \to M_2 \to M_3$ *is exact.*

COROLLARY 2.40. *If* R *is a Noetherian ring, and* M *is finitely locally free* $R-$*module, then* M *is finitely presented.*

PROOF. Using that M is finitely locally free, there are $f_1, \ldots, f_n \in R$ such that $(f_1, \ldots, f_n) = R$ and that each M_{f_i} is finitely free. By Lemmma 2.39 we conclude that M is finitely presented, because a finitely free module is finitely presented. □

PROPOSITION 2.41. *Let* R *be a Noetherian ring, and* $f : R \to A$ *be a homomorphism of rings such that* A *is a locally finitely generated free* $R-$*module. If* $S = \operatorname{End}_R(A)$*, then the functor* $f^*(-) = A \otimes_R -$ *is an equivalence between the categories* $R - \operatorname{Mod}$ *and* $S - \operatorname{Mod}$*, whose inverse is the functor* $H(-) = \operatorname{Hom}_R(A, R) \otimes_S -$.

PROOF. Let $W \subseteq R$ be a multiplicative closed subset, M be an $R-$module and N an $A-$module. Using Corollary 2.40 and Lemma 2.36, we have that

$$W^{-1}S = W^{-1}\operatorname{End}_R(A) = W^{-1}\operatorname{Hom}_R(A, A) \cong \operatorname{Hom}_{W^{-1}R}\left(W^{-1}A, W^{-1}A\right).$$

Using that A is locally finitely free, there are $r_1, \ldots, r_t \in R$ such that $(r_1, \ldots, r_n) = R$ and each $A_{r_i} \cong R_{r_i}^{n_i}$, for some $n_i \in \mathbb{N}$. Note that

Let M be an $R-$module, and N be an $S-$module. As seen in Theorem 2.30, for each r_i there are isomorphisms

$$M_{r_i} \cong \operatorname{Hom}_{R_{r_i}}\left(R_{r_i}^{n_i}, R_{r_i}\right) \otimes_{S_{r_i}} R_{r_i}^{n_i} \otimes_{R_{r_i}} M_{r_i} \cong \left(\operatorname{Hom}_R(A, R) \otimes_S A \otimes_R M\right)_{r_i} \cong \left(Hf^*(M)\right)_{r_i}$$

and

$$N_{r_i} \cong R_{r_i}^{n_i} \otimes_{R_{r_i}} \operatorname{Hom}_{R_{r_i}}\left(R_{r_i}^{n_i}, R_{r_i}\right) \otimes_{S_{r_i}} N_{r_i} \cong \left(A \otimes_R \operatorname{Hom}_R(A, R) \otimes_S N\right)_{r_i} \cong \left(f^*H(N)\right)_{r_i}$$

Using Lemma 2.39, we conclude that F and H are equivalences of categories. □

DEFINITION 2.42. A commutative ring R of characteristic p is said to be $F-$finite if the Frobenius map $F_R : R \to R$ is finite, that is, R is a finite module over $R^p = F_R(R)$.

COROLLARY 2.43 (Frobenius Descent). *Let* R *be regular and* $F-$*finite ring. Consider the Frobenius map* $F^e : R \to {}^eR$*. Then* $(F^e)^*$ *is a equivalence of categories between* $R - \operatorname{Mod}$ *and* $D_R^{(e)} - \operatorname{Mod}$.

PROOF. First note that

$$\operatorname{End}_R\left({}^eR\right) \cong \operatorname{End}_{R^e}(R) = D_R^{(e)}.$$

Since R is regular, ${}^eR$ is locally free. Thus, by Proposition 2.41, we conclude that $(F^e)^*(-) = {}^eR \otimes_R -$ is an equivalence of the categories of $R-$modules and $D_R^{(e)}-$modules, whose inverse is

$$H^e(-) = \operatorname{Hom}_R({}^eR, R) \otimes_{D_R^{(e)}} -.$$

$\square$

We conclude that all $D_R^{(e)}$ are Morita equivalent through R. Additionally, if M is a $D_R^{(e)}-$module, then there is the $R-$module $N = H^e(M)$ such that $(F^e)^*(N) \cong M$. Now, for any $t \in \mathbb{N}$ we consider

$$\left(F^t\right)^*(M) = \left(F^{e+t}\right)^*(N).$$

CHAPTER 3

Generators of D-modules

In this chapter, we consider R as a ring with prime characteristic. Also, in this chapter, we look at generators of R_f as a D_R-module.

LEMMA 3.1. [**9**] *Let R be a regular finitely generated algebra over a regular $F-$finite local ring A of prime characteristic $p > 0$. Then R_f with the D_R-module structure has finite length for every $f \in R$.*

1. Ideal of p-th roots

DEFINITION 3.2. Suppose that R is a free $R^{p^e}-$module. Then any $f \in R$ can be written as

$$f = \sum_{i \in \mathscr{A}} c_i^{p^e} f_i$$

where $\{f_i\}_{i \in \mathscr{A}}$ is an $R^{p^e}-$basis of R. In this context, we define

$$(f)^{[1/p^e]} = (c_1, c_2, \ldots).$$

In this section R is a regular and $F-$finite ring of prime characteristic p.

PROPOSITION 3.3. *The $D_R^{(e)}$ are nested.*

PROOF. Let $e' > e$ and $\delta \in D_R^{(e)}$, then for any $r, f \in R$

$$\delta(r^{p^{e'}} f) = \delta\left(\left(r^{p^{e'-e}}\right)^{p^e} f\right) = \left(r^{p^{e'-e}}\right)^{p^e} \delta(f) = r^{p^{e'}} \delta(f).$$

Thus, in fact $\delta \in D_R^{(e')}$. □

DEFINITION 3.4. Let $I \subseteq R$ be an ideal. We write $I^{[p^e]} = F^e(I)R$, i.e., to the ideal generated by the p^e-th powers of the elements of I. Note that if $\{f_i\}$ is a generating set of I, then $I^{[p^e]} = (\{f_i^{p^e}\})$.

LEMMA 3.5. *If $I, J \subseteq R$ are such that $I^{[p^e]} \subseteq J^{[p^e]}$, then $I \subseteq J$.*

PROOF. If R has no nilpotent elements, then we have that the Frobenius map F is injective. Using the last definition, we have that

$$F^e(I) = F^e(I)R \cap F^e(R) = I^{[p^e]} \cap R^{p^e} \subseteq J^{[p^e]} \cap R^{p^e} = F^e(J)R \cap F^e(R) = F^e(J).$$

Thus, using the injectivity of F, we have that $I \subseteq J$. □

We have defined the ideal generated by the p^e-powers. Now, we would like to define the ideal generated by p^e-roots. One way we can make sense of this is by looking at R as an $R^{p^e}-$module, and then taking the ideal generated by the coefficients of this representation, which is denoted as $I^{[1/p^e]}$.

LEMMA 3.6. *If R is $F-$finite ring. Then*

$$D_R^{(e)} \cdot f = \left((f)^{[1/p^e]}\right)^{[p^e]}.$$

PROOF. Since R is a finitely generated $R^{p^e}-$module and $D_R^{(e)}$ commutes with localization, we may assume that R is a free $R^{p^e}-$module. As we have said before, we can write any $f \in R$ as

$$f = \sum_{i \in A} c_i^{p^e} f_i.$$

It follows that for any $\delta \in D_R^{(e)} = \mathrm{End}_{R^{p^e}}(R)$, then

$$\delta(f) = \sum_{i \in A} c_i^{p^e} \delta(f_i) \in (\{c_i^{p^e}\}) = \left((f)^{[1/p^e]}\right)^{[p^e]},$$

and so $D_R^{(e)} \cdot f \subseteq \left((f)^{[1/p^e]}\right)^{[p^e]}$.

Now, set δ_i as the $R^{p^e}-$linear map defined via $\delta_i(f_j) = \delta_{ij}$, the Kronecker delta. It follows that $\delta_i(f) = c_i^{p^e}$. Thus, we conclude that $\left((f)^{[1/p^e]}\right)^{[p^e]} \subseteq D_R^{(e)} \cdot f$, since each generator of the first is contained in the latter. □

LEMMA 3.7. $(f)^{[1/p^e]} = (f^p)^{\left[1/p^{e+1}\right]}$.

PROOF. It suffices to prove this for each localization at every maximal ideal of R. Thus, we can assume that R is local, with maximal ideal $\mathfrak{m}$, and R is both a free R^p-module and a $R^{p^e}-$module.

Let $\{f_i\}$ be a R^p-basis and $\{g_j\}$ be a $R^{p^e}-$module. Since $1 \notin \mathfrak{m}$, using Nakayama's Lemma, we can take $f_1 = 1$.

For any $r \in R$ there are $a_i \in R$ such that

$$r = \sum_i a_i^p f_i$$

For each a_i there are $b_{ij} \in R$ such that $a_i = \sum_j b_{ij}^{p^e} g_j$. It follows that

$$r = \sum_i \sum_j \left(b_{ij}^{p^e} g_j\right)^p f_i = \sum_{i,j} b_{ij}^{p^{e+1}} g_j^p f_i,$$

and so, $\{g_j^p f_i\}$ is a $R^{p^{e+1}}$basis.

If we write $f = \sum_j c_j^{p^e} g_j$, then, by raising f to the $p-$power, we have that

$$f^p = \sum_j c_j^{p^{e+1}} g_j^p = \sum_j c_j^{p^{e+1}} g_j^p f_1.$$

Thus,

$$(f^p)^{\left[1/p^{e+1}\right]} = (\{c_j\}) = (f)^{[1/p^e]}.|$$

□

LEMMA 3.8. *For any $f, g \in R$, we have that*

$$(fg)^{[1/p^e]} \subseteq (f)^{[1/p^e]} (g)^{[1/p^e]} \subseteq (f)^{[1/p^e]}$$

PROOF. As before, we assume that R is a free $R^{p^e}-$module with a basis $\{f_i\}$. Write $f = \sum_i c_i^{p^e} f_i$ and $g = \sum_i b_i^{p^e} f_i$. Multiplying these, we have

$$fg = \sum_{i,j} c_i^{p^e} b_j^{p^e} f_i f_j = \sum_{i,j,\ell} c_i^{p^e} b_j^{p^e} a_{ij\ell}^{p^e} f_\ell = \sum_{i,j,\ell} (c_i b_j a_{ij\ell})^{p^e} f_\ell,$$

where $a_{ij\ell}$ are such that $f_i f_j = \sum_\ell a_{ij\ell}^{p^e} f_\ell$. Thus, we have that

$$(fg)^{[1/p^e]} = (\{c_i b_j a_{ij\ell}\}) \subseteq (\{c_i\})(\{b_j\}) = (f)^{[1/p^e]} (g)^{[1/p^e]} \subseteq (f)^{[1/p^e]}.$$

The last contention follows using that $(f)^{[1/p^e]}$ is an ideal. $\square$

LEMMA 3.9.

$$\left(f^{p^{e+1}-1}\right)^{[1/p^{e+1}]} \subseteq \left(f^{p^e-1}\right)^{[1/p^e]}.$$

PROOF. Since $f^{p^{e+1}-1} = f^{p^{e+1}-p} f^{p-1}$, we have that $\left(f^{p^{e+1}-1}\right)^{[1/p^{e+1}]} \subseteq \left(f^{p^{e+1}-p}\right)^{[1/p^{e+1}]}$, by Lemma 3.8. Using Lemma 3.7, we conclude that

$$\left(f^{p^{e+1}-1}\right)^{[1/p^{e+1}]} \subseteq \left(f^{p^{e+1}-p}\right)^{[1/p^{e+1}]} = \left((f^{p^e-1})^p\right)^{[1/p^{e+1}]} = \left(f^{p^e-1}\right)^{[1/p^e]}$$

$\square$

2. Ideal of p-th roots and differential operators

PROPOSITION 3.10. *The chain of ideals*

$$\left(f^{p-1}\right)^{[1/p^1]} \supseteq \left(f^{p^2-1}\right)^{[1/p^2]} \supseteq \dots$$

stabilizes at e if and only if there is $\delta \in D_R^{(e+1)}$ such that $\delta\left(\frac{1}{f}\right) = \frac{1}{f^p}$.

PROOF. Suppose the chain stabilizes at e, i.e., $\left(f^{p^e-1}\right)^{[1/p^e]} = \left(f^{p^{e+1}-1}\right)^{[1/p^{e'}]}$ for $e < e'$. Using Lemma 3.7, we have that

$$\left(f^{p^{e+1}-1}\right)^{[1/p^{e+1}]} = \left(f^{p^e-1}\right)^{[1/p^e]} = \left(f^{p^{e+1}-p}\right)^{[1/p^{e+1}]}.$$

It follows from Lemma 3.6 that

$$D_R^{(e+1)} \cdot f^{p^{e+1}-p} = \left(\left(f^{p^{e+1}-p}\right)^{[1/p^{e+1}]}\right)^{[p^{e+1}]} = \left(\left(f^{p^{e+1}-1}\right)^{[1/p^{e+1}]}\right)^{[p^{e+1}]} = D_R^{(e+1)} \cdot f^{p^{e+1}-1}.$$

Since 1_R is R^p-lineal, we have that $f^{p^{e+1}-p} = 1_R \cdot f^{p^{e+1}-p} \in D_R^{(e+1)} \cdot f^{p^{e+1}-1}$, and so there exists $\delta \in D_R^{(e+1)}$ such that

$$f^{p^{e+1}-p} = \delta \cdot f^{p^{e+1}-1} = \delta\left(f^{p^{e+1}-1}\right).$$

Using that δ is $R^{p^{e+1}}-$lineal, we have the equality

$$f^{p^{e+1}-p} = \delta\left(f^{p^{e+1}-1}\right) = f^{p^{e+1}}\delta\left(f^{-1}\right) = f^{p^{e+1}}\delta\left(\frac{1}{f^p}\right),$$

and so

$$\frac{1}{f^p} = \frac{f^{p^{e+1}-p}}{f^{p^{e+1}}} = \delta\left(\frac{1}{f^p}\right).$$

Suppose there is $\delta \in D_R^{(e+1)}$ such that $\delta\left(\frac{1}{f}\right) = \frac{1}{f^p}$. Similarly as in the last paragraph, we obtain that $f^{p^{e+1}-p} = \delta\left(f^{p^{e+1}-1}\right)$. Thus,

$$\begin{aligned}
\left(\left(f^{p^{e+1}-p}\right)^{[1/p^{e+1}]}\right)^{[p^{e+1}]} &= D_R^{(e+1)} \cdot f^{p^{e+1}-p} \\
&= D_R^{(e+1)} \cdot \delta\left(f^{p^{e+1}-1}\right) \\
&= \left(D_R^{(e+1)} \cdot \delta\right) \cdot f^{p^{e+1}-1} \\
&\subseteq D_R^{(e+1)} \cdot f^{p^{e+1}-1} \\
&= \left(\left(f^{p^{e+1}-1}\right)^{[1/p^{e+1}]}\right)^{[p^{e+1}]},
\end{aligned}$$

and $\left(f^{p^{e+1}-p}\right)^{[1/p^{e+1}]} \subseteq \left(f^{p^{e+1}-1}\right)^{[1/p^{e+1}]}$, by using Lemma 3.5. It follows from Lemma 3.7 that the equality

$$\left(f^{p^e-1}\right)^{[1/p^e]} = \left(f^{p^{e+1}-1}\right)^{[1/p^{e+1}]}$$

holds, because $f^{p^{e+1}-p} = \left(f^{p^e-1}\right)^p$.

We showed that if $\delta \in D_R^{(e+1)}$ such that $\delta\left(\frac{1}{f}\right) = \frac{1}{f^p}$, then $\left(f^{p^e-1}\right)^{[1/p^e]} = \left(f^{p^{e+1}-1}\right)^{[1/p^{e+1}]}$. Recall that $D_R^{(e)} \subseteq D_R^{(e')}$ for any $e < e'$, and so, we have that

$$\left(f^{p^e-1}\right)^{[1/p^e]} = \left(f^{p^{e+1}-1}\right)^{[1/p^{e+1}]} = \cdots = \left(f^{p^{e'}-1}\right)^{[1/p^{e'}]}.$$

□

Corollary 3.11. *The chain of ideals*

$$\left(f^{p-1}\right)^{[1/p^1]} \supseteq \left(f^{p^2-1}\right)^{[1/p^2]} \supseteq \dots$$

stabilizes if and only if $\frac{1}{f}$ generates R_f as a D_R module.

Proof. Suppose that $\frac{1}{f}$ generates R_f as a D_R module. Then there is $\delta \in D_R$ such that $\delta(\frac{1}{f}) = \frac{1}{f^p}$. Using Proposition 3.10 we conclude that the chain stabilizes.

Now, suppose that the chain stabilizes at some $e \in \mathbb{N}$. By Proposition 3.10 there is $\delta_1 \in D_R^{(e+1)}$ such that $\delta_1(\frac{1}{f}) = \frac{1}{f^p}$. Additionally, using that R is locally finitely free R^p-module, there is a cover $\{r_1, \dots, r_\ell\}$ such that R_{r_i} is finitely free $R^p_{r_i}-$module. As shown in the proof of Lemma 3.7, we can take a basis that contains 1, and so

$$R_{r_i} \cong R^p_{r_i} \oplus R_{r_i}/R^p_{r_i}.$$

Thus, using Lemma 2.39, we conclude there is a R^p-module isomorphism

$$R_{r_i} \cong R^p \oplus R/R^p.$$

Now, for $x^p + y \in R$, with $x \in R$ and $y \in R/R^p$, define $\delta_2(x^p + y) = \delta_1(x)^p$. We claim that $\delta_2 \in D_R^{(e+2)}$. If $c, r \in R$ with $r = x^p + y$, then

$$\delta_2(c^{p^{e+2}} r) = \delta_2((c^{p^{e+1}} x)^p + c^{p^{e+2}} y) = \delta_1(c^{p^{e+1}} x)^p = \left[c^{p^{e+1}} \delta_1(x)\right]^p = c^{p^{e+2}} \delta_1(x)^p = c^{p^{e+2}} \delta_2(r).$$

Thus, we have that $\delta_2 \in D_R^{(e+2)}$. It follows that

$$\begin{aligned} f^{p^{e+2}} \delta_2\left(\frac{1}{f^p}\right) &= \delta_2\left(f^{p^{e+2}} \frac{1}{f^p}\right) \\ &= \delta_2\left(\left[f^{p^{e+1}} \frac{1}{f}\right]^p\right) \\ &= \left[\delta_1\left(f^{p^{e+1}} \frac{1}{f}\right)\right]^p \\ &= \left[f^{p^{e+1}} \delta_1\left(\frac{1}{f}\right)\right]^p \\ &= \left[f^{p^{e+1}} \frac{1}{f^p}\right]^p \\ &= f^{p^{e+2}} \frac{1}{f^{p^2}}. \end{aligned}$$

Thus, $\delta_2(\frac{1}{f^p}) = \frac{1}{f^{p^2}}$. If we proceed inductively, we have that $\frac{1}{f^{p^e}} \in D_R \cdot \frac{1}{f}$, for every $e \in \mathbb{N}$. Since $\left\{\frac{1}{f^{p^e}}\right\}_{e\in\mathbb{N}}$ is a generating set of R_f as an $R-$module it also generates as an D_R-module. $\square$

THEOREM 3.12. *Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring over a perfect field k of prime characteristic p. If $f \in R$, then the chain $(f^{p-1})^{[1/p^1]} \supseteq \left(f^{p^2-1}\right)^{[1/p^2]} \supseteq \ldots$ stabilizes.*

PROOF. For convenience we will use the multi-index notation, i.e., $\underline{x}^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Note that the monomials $\{\underline{x}^\alpha \mid 0 \le \alpha_i < p^e\}$ is a $R^{p^e}-$basis of R. For any $e \in \mathbb{N}$, there are $c_{e\alpha} \in R$ such that

$$f^{p^e-1} = \sum_\alpha c_{e\alpha}^{p^e} \underline{x}^\alpha.$$

Thus, we have that

$$(p^e - 1)\deg(f) = \deg(f^{p^e-1}) \ge \deg(c_{e\alpha}^{p^e}) = p^e \deg(c_{e\alpha}),$$

and so

$$\deg(f) > \frac{p^e-1}{p^e} \deg(f) \ge \deg(c_{e\alpha}).$$

Since $V = \{q \in R \mid \deg(q) < \deg(f)\}$ is a k vector space of finite dimension, the descending chain

$$V \cap \left(f^{p-1}\right)^{[1/p^1]} \supseteq V \cap \left(f^{p^2-1}\right)^{[1/p^2]} \supseteq \ldots$$

stabilizes. Using that $\{c_{e\alpha}\} \in V \cap \left(f^{p^e-1}\right)^{[1/p^e]}$ is the generating set of $\left(f^{p^e-1}\right)^{[1/p^e]}$, we conclude that the original chain $(f^{p-1})^{[1/p^1]} \supseteq \left(f^{p^2-1}\right)^{[1/p^2]} \supseteq \ldots$ stabilizes. $\square$

Theorem 3.13. *Let R be a regular finitely generated algebra over an $F-$finite regular local ring A of prime characteristic $p > 0$. Let $f \in R$ be a nonzero element. Then $R_f = D_R \cdot \frac{1}{f}$.*

Proof. There is a D_R-isomorphism

$$\theta : F^* R_f \to R_f$$
$$s \otimes \frac{r}{f^t} \mapsto s\frac{r^p}{f^{tp}}.$$

For any $\frac{r}{f^t} \in R_f$ we have that

$$\frac{r}{f^t} = rf^{t(p-1)}\frac{1}{f^{tp}} = \theta\left(rf^{t(p-1)} \otimes \frac{1}{f^t}\right)$$

which shows that θ is surjective. Without loss of generality, we consider that f is not a zero-divisor. Let $s \otimes \frac{r}{f^t} \in \ker(\theta)$, then $s\frac{r^p}{f^{tp}} = 0$ if and only if $sr^p = 0$ if and only if $0 = sr^p \otimes \frac{1}{f^t} = s \otimes \frac{r}{f^t}$, and so θ is injective. Thus, we identified F^*R_f with R_f.

Let $M = D_R \cdot \frac{1}{f}$. We will refer to F^*M as the image under θ. Now, note that

$$\theta\left(f^{p-1} \otimes \frac{1}{f}\right) = f^{p-1}\frac{1}{f^p} = \frac{1}{f} \in F^*M.$$

Since F^*M is a D_R-submodule of R_f that contains $\frac{1}{f}$, we have that $M \subseteq F^*M$. It follows the chain

$$M \subseteq F^*M \subseteq F^{2*}M \subseteq F^{3*}M \subseteq \ldots$$

and that $\frac{1}{f^{p^e}} \in F^{e*}M$, for each $e \in \mathbb{N}$. Thus, $\bigcup_{e\in\mathbb{N}} F^{e*}M = R_f$, since $\left\{\frac{1}{f^{p^s}}\right\}$ is a generating set.

Now, if $M \subsetneq F^*M$, then $F^{e*}M \subsetneq F^{e+1*}M$ for each $e \in \mathbb{N}$, because $F^*(-) = {}^eR \otimes_R -$ and ${}^eR$ is faithfully flat $R-$module. We arrive at a contradiction with Lemma 3.1. Thus, we conclude that

$$M = D_R\frac{1}{f} = R_f.$$

□

3. Differential summands

Direct summands as seen in chapter 2 are useful for describing properties of modules. Furthermore, some families of rings, such as rings of invariants under a linearly reductive group actions, and affine toric rings, are direct summands of polynomial rings. Also, the Hochster-Roberts Theorem [**10**], which states that direct summands of regular rings are Cohen-Macaulay, suggests there are some similar behavior between direct summands and regular rings. We can find such examples where direct summands are normal rings [**11**], with rational singularities in characteristic zero [**12**] or strongly F-regular singularities in prime characteristic [**11**].

Lemma 3.14. *Let $A \subseteq S \subseteq R$ be rings. Let $\iota : S \to R$ be the natural inclusion, and let $\beta \in \operatorname{Hom}_S(R, S)$. If $\delta \in D^i_{S|A}$, then $\beta\delta\iota \in D^i_{S|A}$. Furthermore, the result holds for $\delta \in D^{(i)}_{S|A}$.*

PROOF. Note that $\iota \in \operatorname{Hom}_S(S,R) = D^0_{S|A}(S,R)$ and $\beta \in \operatorname{Hom}_S(R,S) = D^0_{S|A}(R,S)$. Using Proposition 1.16 we have that

$$\delta \in D^i_{R|A}(R,R) \subseteq D^i_{S|A}(R,R).$$

Additionally, with Proposition 1.14, we conclude that

$$\beta\delta\iota \in D^{0+i+0}_{S|A}(S,S) = D^i_{S|A}.$$

For the case $\delta \in D^{(i)}_{S|A}$, we have that

$$\beta\delta\iota(s^{p^i}r) = s^{p^i}\beta\delta\iota(r)$$

since both are $S^{p^0} = S$ lineal. □

THEOREM 3.15. *Let R be a regular $F-$finite domain. Let $S \subseteq R$ be an extension of Noetherian rings such that S is a direct summand of R. Then S_f is generated by $\frac{1}{f}$ as D_S-module.*

PROOF. Recall that R_f is generated as a D_R-module by $\frac{1}{f}$. As seen in the proof of Corollary 3.11, for any $e \in \mathbb{N}$, there and $e' \geq e$ and $\delta \in D^{(e')}_R$ such that $\delta\left(\frac{1}{f}\right) = \frac{1}{f^{p^e}}$. It follows that

$$\delta(f^{p^{e'}-1}) = \delta\left(f^{p^{e'}}\frac{1}{f}\right) = f^{p^{e'}}\delta\left(\frac{1}{f}\right) = f^{p^{e'}}\frac{1}{f^{p^e}} = f^{p^{e'}-p^e}.$$

Let $\beta : R \to S$ be a splitting and $\iota : S \to R$ the inclusion map. Using Lemma 3.14, the map $\hat{\delta} = \beta\delta\iota \in D^{(e')}_S$. Since $p^{e'} - 1 > 0$ and $p^{e'} - p^e > 0$, we have that $f^{p^{e'}-1}, f^{p^{e'}-p^e} \in S$, and so $\iota(f^{p^{e'}-1}) = f^{p^{e'}-1}$ and $\beta(f^{p^{e'}-p^e}) = f^{p^{e'}-p^e}$. We conclude that

$$\hat{\delta}\left(\frac{1}{f}\right) = \frac{1}{f^{p^e}}.$$

Thus, by Corollary 3.11, we conclude that S is generated as a D_S-module by $\frac{1}{f}$. □

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